King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics

MATH 201, Final Exam, Term 173

Duration: 180 minutes

KEY

Name:______________________ ID Number:______________________

Section Number:______________________ Serial Number:______________________

Class Time: ______________________ Instructor's Name:______________________

Instructions:

1. Calculators and Mobiles are not allowed.
2. Write legibly.
3. Show all your work. No points for answers without justification.
4. Make sure that you have 6 pages of problems (Total of 5 written problems)

Question # Number	**Points**	**Maximum Points**
15		18
16		13
17		10
18		14
19		15
Total		70

15. **[18 points]** Let

$$f(x, y) = x^2 + y^2 - 4x$$

(a) Find the critical points of f.

• $f_x(x,y) = 2x - 4$; $f_y(x,y) = 2y$ 1

• $\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \Rightarrow \begin{cases} 2x - 4 = 0 \\ 2y = 0 \end{cases} \Rightarrow \boxed{(x,y) = (2,0).}$ one critical point

1

. f_x & f_y exist at all points (x,y)

(b) Find the local maximum and minimum values and saddle points of f (if any exists)

$f_{xx}(x,y) = 2$, $f_{yy}(x,y) = 2$, $f_{xy}(x,y) = 0$ 1

$D(x,y) = f_{xx}(x,y) \cdot f_{yy}(x,y) - [f_{xy}(x,y)]^2 = 4$

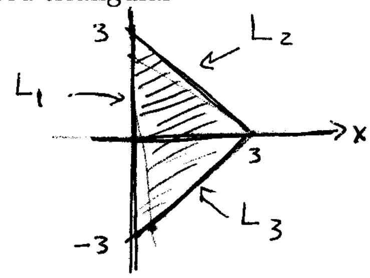
Since $D(2,0) = 4 > 0$ and $f_{xx}(2,0) = 2 > 0$, 1

then f has a local minimum at $(2,0)$ and 1

the local min value is $f(2,0) = -4$.

. There are no local max value & no saddle point of f.

(c) Find the absolute maximum and minimum values of f on the closed triangular region bounded by the curves $x = 0, y = x - 3$, and $y = 3 - x$.



- Critical points of f in the regim

(1) From part (a), there is one critical point in the regim, namely, the point $(2,0)$

- Boundaries

L_1: $x=0$, $-3 \le y \le 3$

(2) $f(0,y) = y^2$

$f'(0,y) = 2y = 0 \Rightarrow y = 0$

points: $(0,0), (0,-3), (0,3)$

L_2: $y = 3 - x$, $0 \le x \le 3$

(1) $f(x, 3-x) = x^2 + (3-x)^2 - 4x$

(1) $f'(x,3-x) = 2x - 2(3-x) - 4 = 4x - 10 = 0 \Rightarrow x = \frac{5}{2}$

(1) points: $(\frac{5}{2}, \frac{1}{2}), (0,3), (3,0)$

L_3: $y = x - 3$, $0 \le x \le 3$

(1) $f(x, x-3) = x^2 + (x-3)^2 - 4x$

(1) $f'(x,x-3) = 2x + 2(x-3) - 4 = 4x - 10 = 0 \Rightarrow x = \frac{5}{2}$

(1) points: $(\frac{5}{2}, -\frac{1}{2}), (0,-3), (3,0)$

(x,y)	$f(x,y)$
$(2,0)$	-4
$(0,0)$	0
$(\frac{5}{2}, \frac{1}{2})$	$-\frac{7}{2}$
$(\frac{5}{2}, -\frac{1}{2})$	$-\frac{7}{2}$
$(0,-3)$	9
$(0,3)$	9
$(3,0)$	-3

← min value of f is -4. (1)

$\}$ ← max value of f is 9 (1)

(2)

16. **[13 points]** Use Lagrange Multipliers to find the absolute maximum and minimum values of

$$f(x,y) = x^2 - xy + y^2$$

on the curve $x^2 + xy + y^2 = 1$. $\Rightarrow g(x,y) = x^2+xy+y^2-1$

- We solve the system :

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ g = 0 \end{cases} \Rightarrow \begin{cases} 2x - y = \lambda\,(2x+y) & \text{—— (1)} \\ -x+2y = \lambda\,(x+2y) & \text{—— (2)} \\ x^2+xy+y^2-1 = 0 & \text{—— (3)} \end{cases}$$

- Case 1 : $2x+y \neq 0$ and $x+2y \neq 0$

$$(1),(2) \Rightarrow \lambda = \frac{2x-y}{2x+y}, \quad \lambda = \frac{-x+2y}{x+2y}$$

$$\Rightarrow \frac{2x-y}{2x+y} = \frac{-x+2y}{x+2y}$$

$$\Rightarrow 2x^2 + \cancel{4xy - xy} - 2y^2 = -2x^2 \cancel{- xy + 4xy} + 2y^2$$

$$\Rightarrow 4x^2 = 4y^2 \Rightarrow y^2 = x^2$$

$$\Rightarrow y = \pm x$$

- $y = x \overset{(3)}{\Longrightarrow} x^2+x^2+x^2-1=0 \Rightarrow x^2 = \frac{1}{3} \Rightarrow x = \pm\frac{1}{\sqrt3}$

points : $\boxed{\left(\frac{1}{\sqrt3}, \frac{1}{\sqrt3}\right), \left(-\frac{1}{\sqrt3}, -\frac{1}{\sqrt3}\right)}$

- $y = -x \overset{(3)}{\Longrightarrow} x^2 - x^2 + x^2 - 1 = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$

points : $\boxed{(-1,1), (1,-1)}$

- Case 2 : $2x+y=0$ or $x+2y=0$
  - If $2x+y=0$, then (1) implies $2x-y=0$. Solving gives $(x,y)=(0,0)$, rejected as it does not satisfy (3).
  - If $x+2y=0$, then (2) implies $-x+2y=0$. Solving gives $(x,y)=(0,0)$, rejected as it does not satisfy (3).

(x,y)	$f(x,y)$
$\left(\frac{1}{\sqrt3}, \frac{1}{\sqrt3}\right)$	$\frac{1}{3}$
$\left(-\frac{1}{\sqrt3}, -\frac{1}{\sqrt3}\right)$	$\frac{1}{3}$
$(-1,1)$	3
$(1,-1)$	3

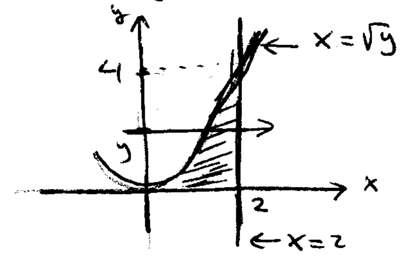
$\leftarrow$ min. value of f is $\frac{1}{3}$

$\leftarrow$ max value of f is 3

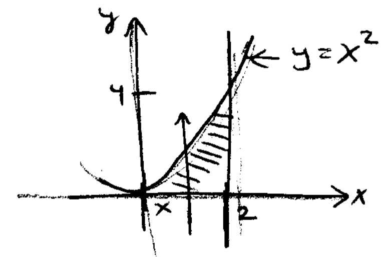
17. **[10 points]** Evaluate $\int_0^4 \int_{\sqrt{y}}^2 \cos(x^3)\, dx\, dy$

To be able to integrate, Reverse the order of integration

$R: \sqrt{y} \le x \le 2,\ 0 \le y \le 4$ (Type II)



$R: 0 \le x \le 2,\ 0 \le y \le x^2$ (Type I)



$$\int_0^4 \int_{\sqrt{y}}^2 \cos(x^3)\, dx\, dy = \int_0^2 \int_0^{x^2} \cos(x^3)\, dy\, dx \qquad (3)+(3)$$

$$= \int_0^2 \cos(x^3) \cdot y \Big]_{y=0}^{y=x^2} dx \qquad (1)$$

$$= \int_0^2 \cos(x^3) \cdot x^2\, dx \qquad (1)$$

$$= \frac{1}{3} \sin(x^3) \Big]_0^2 \qquad (1)$$

$$= \frac{1}{3} \sin(8). \qquad (1)$$

18. [**14 points**] Change the integral to cylindrical coordinates and then evaluate:

$$I = \int_0^4 \int_{-\sqrt{16-y^2}}^0 \int_{\sqrt{x^2+y^2}}^4 (x^2+y^2)\, dz\, dx\, dy$$

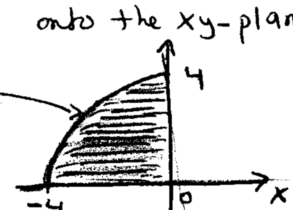
- The triple Integral is over the Solid region

$$E = \{ (x,y,z) : \underbrace{0 \le y \le 4,\ -\sqrt{16-y^2} \le x \le 0}_{R:\ \text{the projection of } E \text{ onto the } xy\text{-plane}},\ \sqrt{x^2+y^2} \le z \le 4 \}$$

$x = -\sqrt{16-y^2}$

$\Rightarrow x^2+y^2 = 16$

$\Rightarrow r = 4$



In polar Coord, R : $0 \le r \le 4$, $\frac{\pi}{2} \le \theta \le \pi$

$z = \sqrt{x^2+y^2}$, upper surface of E (upper Cone)

$z = 4$, Lower surface of E (a plane)

$r \le z \le 4$

- In Cylindrical Coord,

$E = \{ (r, \theta, z) : 0 \le r \le 4,\ \frac{\pi}{2} \le \theta \le \pi,\ r \le z \le 4 \}$ (2) + (2) + (2)

- So

$$I = \int_{\frac{\pi}{2}}^{\pi} \int_0^4 \underbrace{\int_r^4 \underset{(1)}{r^2}\, dz}_{r^2 \cdot z \big]_{z=r}^{z=4}} \cdot \underset{(1)}{r\, dr\, d\theta}$$

$$= \int_{\frac{\pi}{2}}^{\pi} \int_0^4 r^2(4-r) \cdot r\, dr\, d\theta \quad (2)$$

$$= \int_{\frac{\pi}{2}}^{\pi} \int_0^4 (4r^3 - r^4)\, dr\, d\theta$$

$$= \int_{\frac{\pi}{2}}^{\pi} \left(r^4 - \frac{r^5}{5}\right)\Big]_{r=0}^{r=4} d\theta$$

$$= \left(4^4 - \frac{4^5}{5}\right) \int_{\frac{\pi}{2}}^{\pi} d\theta \quad (2)$$

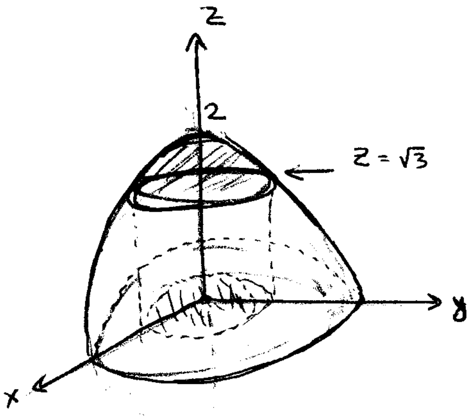
$$= 4^4\left(1 - \frac{4}{5}\right) \cdot \theta\Big]_{\frac{\pi}{2}}^{\pi} = \frac{4^4}{5} \cdot \frac{\pi}{2}$$

$$= \frac{128}{5}\pi \quad (2)$$

19. **[15 points]** Let $E = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 4,\ z \geq \sqrt{3}\}$.

(a) Describe the set E in spherical coordinates.

(b) Evaluate $\iiint_E z\, dV$.



(a) : $x^2+y^2+z^2 = 4 \Rightarrow \rho^2 = 4 \Rightarrow \rho = 2$ (as $\rho \geq 0$) (1)

$z = \sqrt{3} \Rightarrow \rho \cos\phi = \sqrt{3} \Rightarrow \rho = \sqrt{3}\sec\phi$ (1)

. Intersection: $2 = \sqrt{3}\sec\phi$

$\Rightarrow \cos\phi = \frac{\sqrt{3}}{2} \Rightarrow \phi = \frac{\pi}{6}$ (1)

. The projection of E onto the xy-plane is the disk $x^2+y^2 \leq 1$

$$E = \{(\rho, \theta, \phi) : \underset{(2)}{\sqrt{3}\sec\phi \leq \rho \leq 2},\ \underset{(1)}{0 \leq \theta \leq 2\pi},\ \underset{(1)}{0 \leq \phi \leq \frac{\pi}{6}}\}$$

(b) $$\iiint_E z\, dV = \int_0^{2\pi}\int_0^{\pi/6}\int_{\sqrt{3}\sec\phi}^{2} \underset{(1)}{\rho\cos\phi} \cdot \underset{(1)}{\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta}$$

$$= \int_0^{2\pi}\int_0^{\pi/6} \cos\phi \sin\phi \cdot \frac{1}{4}\rho^4\Big]_{\rho=\sqrt{3}\sec\phi}^{\rho=2} d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/6} \cos\phi\sin\phi \cdot \frac{1}{4}(16 - 9\sec^4\phi)\, d\phi\, d\theta \quad (1)$$

$$= \int_0^{2\pi}\int_0^{\pi/6} 4\sin\phi\cos\phi - \frac{9}{4}\frac{\sin\phi}{\cos^3\phi}\, d\phi\, d\theta$$

$$= \int_0^{2\pi} \left(2\sin^2\phi - \frac{9}{8}\frac{1}{\cos^2\phi}\right)\Big]_{\phi=0}^{\phi=\frac{\pi}{6}} d\theta \quad (1) + (2)$$

$$\left(\frac{1}{2} - \frac{9}{8}\cdot\frac{4}{3}\right) - \left(0 - \frac{9}{8}\right) = \frac{1}{2} - \frac{3}{2} + \frac{9}{8} = \frac{1}{8} \quad (1)$$

$$= \frac{1}{8}\int_0^{2\pi} d\theta$$

$$= \frac{1}{8} \cdot \theta\Big]_0^{2\pi}$$

$$= \frac{\pi}{4} \quad (1)$$

1. At the point $(x, y) = (0, 0)$, the parametric curve

$$x = \cos t,\ y = \sin t \cos t,\ 0 < t < 2\pi$$

has (slant= neither horizontal nor vertical)

$x = \cos t = 0 \Rightarrow t = \frac{\pi}{2}, \frac{3\pi}{2}$

$y = \sin t \cos t = 0 \Rightarrow t = \pi, \frac{\pi}{2}, \frac{3\pi}{2}$

We choose the common values:

$\boxed{t = \frac{\pi}{2}, \frac{3\pi}{2}}$

(a) two slant tangent lines

(b) one slant tangent line

(c) one horizontal tangent line

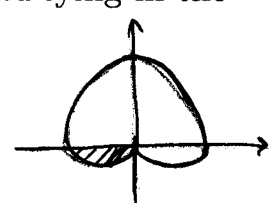
(d) one vertical tangent line

(e) two horizontal tangent lines

$\frac{dy}{dx} = \frac{\cos(2t)}{-\sin t} = \quad t=\frac{\pi}{2} \to 1, \quad t=\frac{3\pi}{2} \to -1$

$\Rightarrow$ two slant tangent lines

2. The **area** of the region **inside** the cardioid $r = 1 + \sin\theta$ and lying in the **third quadrant** is equal to



(a) $\frac{3\pi}{8} - 1$

(b) $\frac{\pi}{4} + 2$

(c) $\frac{5\pi}{4} + 1$

(d) $\frac{5\pi}{8} - 2$

(e) $\pi + \frac{1}{2}$

$A = \int_{\pi}^{3\pi/2} \frac{1}{2} (1 + \sin\theta)^2 \, d\theta$

$= \frac{1}{2} \int_{\pi}^{3\pi/2} 1 + 2\sin\theta + \frac{1}{2} - \frac{1}{2}\cos(2\theta) \, d\theta$

$= \frac{1}{2} \int_{\pi}^{3\pi/2} \frac{3}{2} + 2\sin\theta - \frac{1}{2}\cos(2\theta) \, d\theta$

$= \frac{1}{2} \left[\frac{3}{2}\theta - 2\cos\theta - \frac{1}{4}\sin(2\theta) \right]_{\pi}^{3\pi/2}$

$= \frac{1}{2} \left[\left(\frac{9\pi}{4} - 0 - 0\right) - \left(\frac{3\pi}{2} + 2 - 0\right) \right]$

$= \frac{1}{2} \left(\frac{9\pi}{4} - \frac{6\pi}{4} - 2 \right)$

$= \frac{1}{2} \left(\frac{3\pi}{4} - 2 \right)$

$= \frac{3\pi}{8} - 1$

3. Let $\vec{u} = < 1, 1 >$ and $\vec{v} = < 1, -1 >$. If $\vec{w}$ is a vector such that $\langle w_1, w_2 \rangle$

$$\text{comp}_{\vec{u}}\vec{w} = -1, \quad \text{comp}_{\vec{v}}\vec{w} = 2$$

then $\vec{w} =$

$\Rightarrow$ • $\frac{\vec{u}\cdot\vec{w}}{|\vec{u}|} = -1 \Longrightarrow w_1 + w_2 = -\sqrt{2}$

• $\frac{\vec{v}\cdot\vec{w}}{|\vec{v}|} = 2 \Longrightarrow w_1 - w_2 = 2\sqrt{2}$

Solving the system, we get

$w_1 = \frac{\sqrt{2}}{2}, \quad w_2 = -\frac{3\sqrt{2}}{2}$

So $\vec{w} = \langle \frac{\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2} \rangle$

$= \frac{1}{2} \langle \sqrt{2}, -3\sqrt{2} \rangle$

(a) $\frac{1}{2} < \sqrt{2}, -3\sqrt{2} >$

(b) $\frac{1}{2} < -\sqrt{2}, 3\sqrt{2} >$

(c) $\frac{1}{2} < -\sqrt{2}, -3\sqrt{2} >$

(d) $\frac{1}{2} < 3\sqrt{2}, -\sqrt{2} >$

(e) $\frac{1}{2} < -3\sqrt{2}, -\sqrt{2} >$

4. The set $\{(x, y, z) \in R^3 : x^2 + y^2 + z^2 > 2z\}$ represents the region

(a) outside the sphere with center $(0, 0, 1)$ and radius 1

(b) inside the sphere with center $(0, 0, 1)$ and radius 1

(c) outside the sphere with center $(0, 0, 0)$ and radius $\sqrt{2}$

(d) outside and on the sphere with center $(0, 0, 1)$ and radius 1

(e) outside the sphere with center $(0, 0, 0)$ and ~~center~~ radius 1.

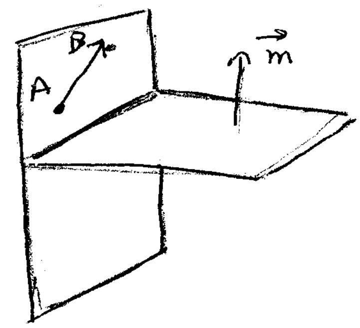
$x^2 + y^2 + z^2 - 2z > 0$

$x^2 + y^2 + z^2 - 2z + 1 > 0 + 1$

$x^2 + y^2 + (z-1)^2 > 1$

5. If $ax + by + 7z = d$ is the equation of the plane that passes through the points $A(0,-2,1)$ and $B(-1,3,2)$ and is perpendicular to the plane $x + 2y - 2z = 7$, then $abd =$

$\hookrightarrow \vec{m} = \langle 1, 2, -2 \rangle$



(a) 60

(b) -40

(c) -10

(d) 35

(e) 23

. $\vec{AB} = \langle -1, 5, 1 \rangle$

. $\vec{n} = \vec{AB} \times \vec{m}$

$$= \begin{vmatrix} i & j & k \\ -1 & 5 & 1 \\ 1 & 2 & -2 \end{vmatrix}$$

$$= \langle -12, -1, -7 \rangle$$

, Eq is:

$$-12(x-0) - 1(y+2) - 7(z-1) = 0$$
$$-12x - y - 2 - 7z + 7 = 0$$
$$-12x - y - 7z = -5$$
$$12x + y + 7z = 5$$

$a = 12,\ b = 1,\ d = 5 \implies abd = 60$

6. Which one of the following statements is **False** about the surface $x^2 - 3y^2 - 4z^2 - 2x - 12y - 10 = 0$.

Completing the square, we get $-(x-1)^2 + 3(y+2)^2 + 4z^2 = 1$

a hyperboloid of one sheet

(a) it is a hyperboloid of two sheets

(b) the axis of the surface is parallel to the x − axis

(c) the trace of the surface in the plane $z = \dfrac{1}{2}$ is two lines

(d) the trace of the surface in the plane $z = 0$ is a hyperbola

(e) the trace of the surface in the plane $x = 1$ is an ellipse

7. $\lim\limits_{(x,y)\to(0,0)} \dfrac{3x^2y^2}{x^4+y^4}$

(a) does not exist

(b) is equal to 3

(c) is equal to $\frac{1}{2}$

(d) is equal to 1

(e) is equal to 0

• along the x-axis : $y=0$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=0}} \frac{3x^2y^2}{x^4+y^4} = \lim_{x\to 0} 0 = 0$$

• along the line $y=x$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=x}} \frac{3x^2y^2}{x^4+y^4} = \lim_{x\to 0} \frac{3}{2} = \frac{3}{2}$$

• Since the two limits are not equal, then the given limit does not exist.

8. If $f(x,y,z) = \dfrac{xy^2}{z^3} + \sin^3(y^2\sqrt{z})$, then $f_{zyx}(3,-2,2) =$

(a) $\frac{3}{4}$

(b) $-\frac{3}{8}$

(c) $-\frac{2}{5}$

(d) 2

(e) $-\frac{1}{4}$

• Using Clairaut's Theorem, we can change the order of differentiation (and hence simplify the calculation):

$$f_{zyx}(3,-2,2) = f_{xyz}(3,-2,2).$$

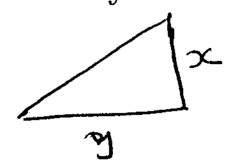
• $f_x(x,y,z) = \dfrac{y^2}{z^3} + 0 = y^2 z^{-3}$

$f_{xy}(x,y,z) = 2y z^{-3}$

$f_{xyz}(x,y,z) = -6y z^{-4} = \dfrac{-6y}{z^4}$

• $f_{xyz}(3,-2,2) = \dfrac{-6(-2)}{2^4} = \dfrac{12}{16} = \dfrac{3}{4}$

9. The two legs of a right triangle are measured as $5\,cm$ and $12\,cm$ with a possible error in measurement of at most $0.2\,cm$ in each. The **maximum error** in the calculated value of the area of the triangle is approximately equal to
(**Hint**: Use differentials)



- $x = 5,\ y = 12,\quad |\Delta x| \le 0.2$, $|\Delta y| \le 0.2$

(a) $1.7\,cm^2$

(b) $0.4\,cm^2$

(c) $1\,cm^2$

(d) $1.2\,cm^2$

(e) $0.8\,cm^2$

- $A = \frac{1}{2}xy \Rightarrow dA = A_x\,dx + A_y\,dy$
$= \frac{1}{2}y\,dx + \frac{1}{2}x\,dy$
$\Rightarrow dA = \frac{1}{2}\cdot 12\cdot(0.2) + \frac{1}{2}5\cdot(0.2)$
$= 1.2 + 0.5$
$= 1.7$

- $\Delta A \approx dA = 1.7\ cm^2$

10. The **directional derivative** of $f(x,y) = x\sqrt{y}$ at $(2,4)$ in the direction of the vector $\vec{v} = <2,-1>$ is

(a) $\frac{7\sqrt{5}}{10}$

(b) $-\frac{\sqrt{5}}{5}$

(c) $-\frac{2\sqrt{5}}{5}$

(d) $\frac{3\sqrt{5}}{10}$

(e) $\frac{2\sqrt{5}}{5}$

- $\nabla f(x,y) = \langle f_x, f_y\rangle = \langle \sqrt{y}, \frac{x}{2\sqrt{y}}\rangle$
$\nabla f(2,4) = \langle 2, \frac{1}{2}\rangle$
- $\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{1}{\sqrt{5}}\langle 2,-1\rangle = \langle \frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}}\rangle$

- $D_{\vec{v}}f(2,4) = D_{\vec{u}}f(2,4)$
$= \nabla f(2,4)\cdot\vec{u}$
$= \langle 2, \frac{1}{2}\rangle\cdot\langle\frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}}\rangle$
$= \frac{4}{\sqrt{5}} - \frac{1}{2\sqrt{5}}$
$= \frac{7}{2\sqrt{5}} = \frac{7\sqrt{5}}{10}$

11. An equation for the **tangent plane** to the surface $x^2y + e^{zx} = 3$ at the point $(1, 2, 0)$ is

$F(x,y,z) = x^2y + e^{xz} - 3$

$\nabla F(x,y,z) = \langle F_x, F_y, F_z \rangle$

$= \langle 2xy + e^{xz}\cdot z,\ x^2,\ e^{xz}\cdot x \rangle$

$\vec{n} = \nabla F(1,2,0) = \langle 4, 1, 1 \rangle$

eq:

$4(x-1) + 1(y-2) + 1(z-0) = 0$

$\Longrightarrow 4x - 4 + y - 2 + z = 0$

$\Longrightarrow 4x + y + z = 6$

(a) $4x + y + z = 6$

(b) $3x - y + z = 1$

(c) $-2x + 3y + 4z = 4$

(d) $2x + y + z = 4$

(e) $-3x + 4y - z = 5$

$\begin{cases} f_x(x,y) = 6xy - 12x = 0 \\ f_y(x,y) = 3y^2 - 12y + 3x^2 = 0 \end{cases} \Longrightarrow \begin{cases} xy - 2x = 0 & (1) \\ y^2 - 4y + x^2 = 0 & (2) \end{cases}$

$(1) \Longrightarrow x(y-2) = 0 \Longrightarrow x = 0,\ y = 2$

12. The function $f(x,y) = y^3 - 6y^2 + 3x^2y - 6x^2$ has

• $x = 0 \overset{(2)}{\Longrightarrow} y^2 - 4y = 0 \Longrightarrow y = 0, y = 4$

$\boxed{(x,y) = (0,0), (0,4)}$

• $y = 2 \overset{(2)}{\Longrightarrow} 4 - 8 + x^2 = 0 \Longrightarrow x^2 = 4 \Longrightarrow x = \pm 2$

$\boxed{(x,y) = (-2,2), (2,2)}$

(a) two saddle points

(b) two local maxima

(c) two local minima and one saddle point

(d) three local minima

(e) three critical points only

Test:

$f_{xx}(x,y) = 6y - 12$

$f_{yy}(x,y) = 6y - 12$

$f_{xy}(x,y) = 6x$

$D(x,y) = f_{xx}f_{yy} - (f_{xy})^2$

$= (6y-12)^2 - (6x)^2$

$= 36[(y-2)^2 - x^2]$

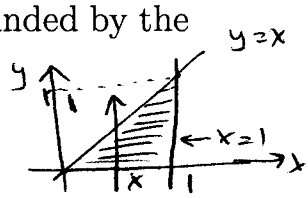
(0,0): $D(0,0) = 36\cdot 4 > 0$, $f_{xx}(0,0) = -12 < 0 \Longrightarrow$ Local max at $(0,0)$

(0,4): $D(0,4) = 36\cdot 4 > 0$, $f_{xx}(0,4) = 24 - 12 = 12 > 0 \Longrightarrow$ Local min at $(0,4)$

(-2,2): $D(-2,2) = 36(-4) < 0 \Longrightarrow$ Saddle pt at $(-2,2)$

D(2,2): $D(2,2) = 36(-4) < 0 \Longrightarrow$ Saddle pt at $(2,2)$

13. The **average value** of $f(x,y) = \dfrac{y}{1+x^2}$ over the region R bounded by the curves $y=0, y=x, x=1$ is equal to



(a) $\dfrac{4-\pi}{4}$

(b) π

(c) $\dfrac{\pi-2}{8}$

(d) $\dfrac{1}{2}$

(e) $\dfrac{\pi}{2}$

$$f_{ave} = \frac{1}{\text{area}(R)} \iint_R f(x,y)\, dA$$

area $(R) = \frac{1}{2}\cdot 1\cdot 1 = \frac{1}{2}$

R: Type I

$0 \le x \le 1$

$0 \le y \le x$

$$= 2\int_0^1 \int_0^x \frac{y}{1+x^2}\, dy\, dx$$

$$= 2\int_0^1 \frac{1}{1+x^2}\cdot \frac{1}{2} y^2 \Big]_{y=0}^{y=x} dx$$

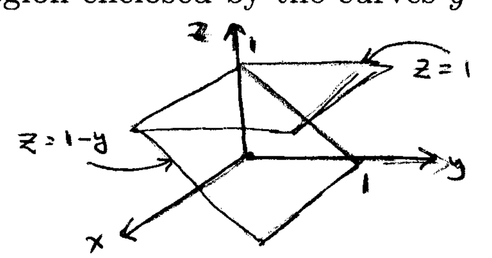
$$= \int_0^1 \frac{x^2}{1+x^2}\, dx$$

$$= \int_0^1 1 - \frac{1}{1+x^2}\, dx$$

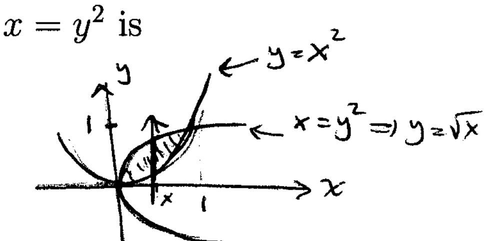
$$= x - \tan^{-1} x \Big]_0^1$$

$$= \left(1 - \frac{\pi}{4}\right) - (0-0) = \frac{4-\pi}{4}$$

14. The **volume** of the solid between the planes $z=1$ and $y+z=1$ and above the region enclosed by the curves $y=x^2$ and $x=y^2$ is



$\xrightarrow{R}$



(a) $\dfrac{3}{20}$

(b) $\dfrac{4}{5}$

(c) $\dfrac{1}{2}$

(d) $\dfrac{2}{9}$

(e) $\dfrac{5}{8}$

Type R: $0 \le x \le 1$

$x^2 \le y \le \sqrt{x}$

$1-y \le z \le 1$

$V = \iiint_D dV$, D is the Solid

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} \int_{1-y}^1 dz\, dy\, dx$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} z \Big]_{z=1-y}^{z=1} dy\, dx = \int_0^1 \int_{x^2}^{\sqrt{x}} y\, dy\, dx$$

$$= \int_0^1 \frac{1}{2} y^2 \Big]_{y=x^2}^{y=\sqrt{x}} dx = \frac{1}{2}\int_0^1 (x - x^4)\, dx$$

$$= \frac{1}{2}\cdot\left(\frac{1}{2}x^2 - \frac{1}{5}x^5\right)\Big]_0^1 = \frac{1}{2}\cdot\left(\frac{1}{2}-\frac{1}{5}\right) = \frac{1}{2}\cdot\frac{3}{10}$$