King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics

MATH 201, Exam II, Term 173

Duration: 120 minutes

the KEY SOLUTION

Name: ____________ ID Number: ____________

Section Number: ____________ Serial Number: ____________

Class Time: ____________ Instructor's Name: ____________

Instructions:

1. Calculators and Mobiles are not allowed.
2. Write legibly.
3. Show all your work. No points for answers without justification.
4. Make sure that you have 7 pages of problems (Total of 7 Problems)

Question # Number	Points	Maximum Points
1		14
2		15
3		10
4		13
5		18
6		18
7		12
Total		100

1. **[4+10 points]** Consider the following planes

$$\mathfrak{p}_1 : 2x + y - z = 2$$

$$\mathfrak{p}_2 : x - y + z = 1$$

(a) Find the distance from the point $(2, -3, 5)$ to the plane $\mathfrak{p}_1$.

$\mathfrak{p}_1 : \; 2x + y - z - 2 = 0$

$$d = \frac{|2(2) + (-3) - 5 - 2|}{\sqrt{(2)^2 + (1)^2 + (-1)^2}} \qquad (2)$$

$$= \frac{6}{\sqrt{6}} \qquad (2)$$

or $= \sqrt{6}$

(b) Find parametric equations for the line of intersection of the planes $\mathfrak{p}_1$ and $\mathfrak{p}_2$.

(1) + (1) • $\mathfrak{p}_1 \Rightarrow \vec{n}_1 = \langle 2, 1, -1 \rangle$; $\mathfrak{p}_2 \Rightarrow \vec{n}_2 = \langle 1, -1, 1 \rangle$

• a vector parallel to the line of intersection is

$$\vec{v} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix} = \langle 0, -3, -3 \rangle \quad (1)$$

(2)

• a point on the line of intersection :

Set $y = 0 \Rightarrow \begin{cases} 2x - z = 2 \\ x + z = 1 \end{cases} \Rightarrow x = 1, z = 0 \Rightarrow (x, y, z) = (1, 0, 0)$

(2)

- Parametric equations for the line of intersection :

(3)
$$x = 1$$
$$y = -3t \qquad , t \in \mathbb{R}$$
$$z = -3t$$

2. **[4+4+7 points]** Consider the surface whose equation is given by

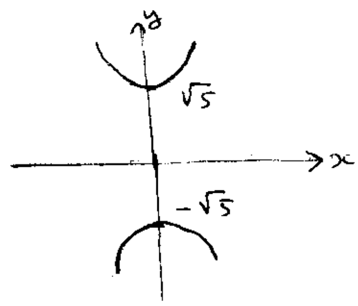
$$5x^2 - y^2 + z^2 - 4z + 5 = 0$$

(a) Find an equation for the trace of the surface in the xy-plane. Sketch the resulting equation.

Set $z = 0$. The equation for the trace is

$$5x^2 - y^2 + 5 = 0 \qquad (2)$$

$$\text{or} \quad y^2 - 5x^2 = 5$$



, a hyperbola

(2)

(b) Write the equation of the surface in standard form.

$$5x^2 - y^2 + z^2 - 4z + 4 = -5 + 4$$

(2) $$5x^2 - y^2 + (z-2)^2 = -1$$

(2) $$-5x^2 + y^2 - (z-2)^2 = 1$$

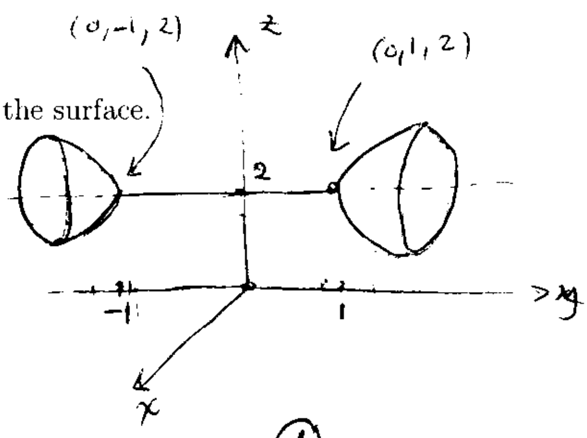
$$\text{or} \quad -\frac{x^2}{(1/\sqrt{5})^2} + y^2 - (z-2)^2 = 1$$

(c) Identify (name, vertex(ces), axis) and sketch the surface.

name : a hyperboloid of two sheets (1)

Verteces : $(0,-1,2)$, $(0,1,2)$ (1)

axis : the line through the verteces (1)
& parallel to the y-axis.

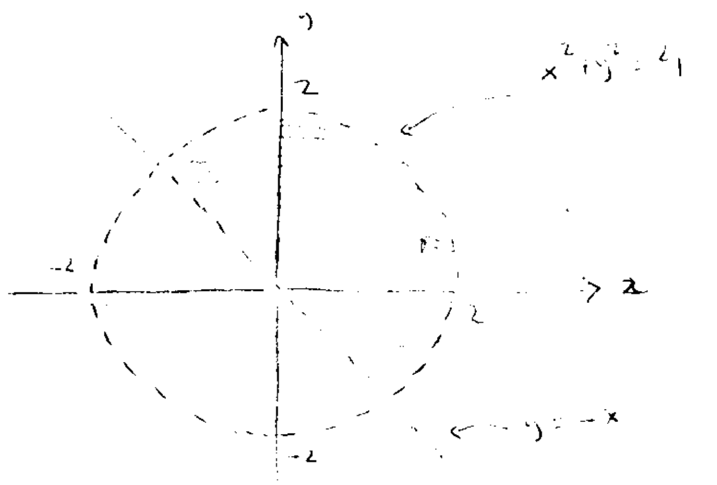


(4)

3. [10 **points**] Find and sketch the domain of $f(x,y) = \dfrac{\ln(4 - x^2 - y^2)}{x+y}$.

Domain $= \{ (x,y) \in \mathbb{R}^2 : 4 - x^2 - y^2 > 0$ and $x + y \neq 0 \}$ 3 + 2

$= \{ (x,y) \in \mathbb{R}^2 : x^2 + y^2 < 4$ and $y \neq -x \}$



4. **[6+7 points]** Find the limit, if it exists, or show that the limit does not exist.

(a) $\lim_{(x,y,z)\to(1,2,1)} \dfrac{z^3 - xz^2 + zy - xy}{\sqrt{x+y} - \sqrt{z+y}}$ $\longrightarrow z^2(z-x) + y(z-x)$

$= \lim_{(x,y,z)\to(1,2,1)} \dfrac{(z^2+y)(z-x)}{\sqrt{x+y} - \sqrt{z+y}}$ ②

$= \lim_{(x,y,z)\to(1,2,1)} \dfrac{(z^2+y)(z-x)}{\sqrt{x+y} - \sqrt{z+y}} \cdot \dfrac{\sqrt{x+y} + \sqrt{z+y}}{\sqrt{x+y} + \sqrt{z+y}}$ ①

$= \lim_{(x,y,z)\to(1,2,1)} \dfrac{(z^2+y)(z-x)}{(x-z)} \cdot (\sqrt{x+y} + \sqrt{z+y})$ ①

$= \lim_{(x,y,z)\to(1,2,1)} -(z^2+y) \cdot (\sqrt{x+y} + \sqrt{z+y})$ ①

$= -(1+2)(\sqrt{3} + \sqrt{3}) = -6\sqrt{3}$ ①

(b) $\lim_{(x,y)\to(0,0)} \dfrac{1 - \cos y}{x^2 + y^2}$

③ • along the x-axis: $y=0$

$\lim_{\substack{(x,y)\to(0,0)\\ y=0}} \dfrac{1-\cos y}{x^2+y^2} = \lim_{x\to 0} \dfrac{1-\cos 0}{x^2+0^2} = \lim_{x\to 0} \dfrac{0}{x^2} = \lim_{x\to 0} 0 = 0$

③ • along the y-axis: $x=0$

$\lim_{\substack{(x,y)\to(0,0)\\ x=0}} \dfrac{1-\cos y}{x^2+y^2} = \lim_{y\to 0} \dfrac{1-\cos y}{0^2+y^2} = \lim_{y\to 0} \dfrac{1-\cos y}{y^2}$

$\overset{L'R}{=} \lim_{y\to 0} \dfrac{\sin y}{2y} \overset{L'R}{=} \lim_{y\to 0} \dfrac{\cos y}{2} = \dfrac{1}{2}$

① • Since the two limits along the above two paths are <u>not</u> equal, then

$\lim_{(x,y)\to(0,0)} \dfrac{1-\cos y}{x^2+y^2}$ does not exist

5. [5+6+7 points]

(a) Let $u(x,y) = \ln(x^2+y^2)$. Find $u_{xx}(x,y) + u_{yy}(x,y)$.

① + ① $\quad \cdot\ u_x(x,y) = \dfrac{2x}{x^2+y^2} \Longrightarrow u_{xx}(x,y) = \dfrac{(x^2+y^2)\cdot 2 - 2x(2x)}{(x^2+y^2)^2} = \dfrac{2y^2-2x^2}{(x^2+y^2)}$

① + ① $\quad \cdot\ u_y(x,y) = \dfrac{2y}{x^2+y^2} \Longrightarrow u_{yy}(x,y) = \dfrac{(x^2+y^2)\cdot 2 - 2y(2y)}{(x^2+y^2)^2} = \dfrac{2x^2-2y^2}{(x^2+y^2)^2}$

$\cdot\ u_{xx}(x,y) + u_{yy}(x,y) = \dfrac{2y^2-2x^2}{(x^2+y^2)^2} + \dfrac{2x^2-2y^2}{(x^2+y^2)^2}$

② $\quad = 0$

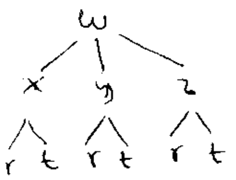
(b) The equation $e^{xz} = 2\sin^2(yz)$ defines z implicitly as a differentiable function of x and y. Compute $\dfrac{\partial z}{\partial y}$ at the point $(x,y,z) = \left(0, \dfrac{\pi}{4}, 1\right)$.

Let $F(x,y,z) = e^{xz} - 2\sin^2(yz)$ ①

$\dfrac{\partial z}{\partial y} = -\dfrac{F_y}{F_z}$ ②

$= -\dfrac{0 - 2\cdot 2\sin(yz)\cdot \cos(yz)\cdot z}{e^{xz}\cdot x - 2\cdot 2\sin(yz)\cos(yz)\cdot y}$ $\longrightarrow$ ① $\longrightarrow$ ①

$\left.\dfrac{\partial z}{\partial y}\right|_{(0,\frac{\pi}{4},1)} = -\dfrac{-4\,\frac{\sqrt2}{2}\cdot\frac{\sqrt2}{2}\cdot 1}{0 - 4\cdot\frac{\sqrt2}{2}\cdot\frac{\sqrt2}{2}\cdot\frac{\pi}{4}} = -\dfrac{4}{\pi}$ ①



(c) Let $w = \frac{1}{4}x^2 + y\tan^{-1}(2z)$, $x = \frac{r}{t}$, $y = r^2t^3$, $z = 2r+t$. Use the chain rule to compute $\dfrac{\partial w}{\partial t}$ when $r = \dfrac{1}{2}$, $t = -\dfrac{1}{2}$. ①

$\rightarrow\ x = -1,\ y = -\frac{1}{32},\ z = \frac{1}{2}$

$\dfrac{\partial w}{\partial t} = \dfrac{\partial w}{\partial x}\cdot\dfrac{\partial x}{\partial t} + \dfrac{\partial w}{\partial y}\cdot\dfrac{\partial y}{\partial t} + \dfrac{\partial w}{\partial z}\cdot\dfrac{\partial z}{\partial t}$ ②

$= \frac{1}{2}x\cdot\left(-\frac{r}{t^2}\right) + \tan^{-1}(2z)\cdot(3r^2t^2) + y\cdot\dfrac{2}{1+4z^2}\cdot(1)$ ①+①+①

$\left.\dfrac{\partial w}{\partial t}\right|_{r=\frac12,\ t=-\frac12} = \left(-\frac{1}{2}\right)\cdot(-2) + \frac{\pi}{4}\cdot\frac{3}{16} - \frac{1}{32} = \frac{31}{32} + \frac{3\pi}{64}$

or $= \dfrac{62+3\pi}{64}$ ①

6. **[6+12 points]**

(a) Find an equation for the tangent plane to the surface $z = x\,e^{xy}$ at the point $(2, 0, 2)$.

$Z = x e^{xy} = f(x,y)$

The equation is

② $Z = f(2,0) + f_x(2,0)(x-2) + f_y(2,0)(y-0)$

$\cdot\ f(2,0) = 2$ ①

$\cdot\ f_x(x,y) = x e^{xy} \cdot y + e^{xy} \cdot 1 \implies f_x(2,0) = 1$ ①

$\cdot\ f_y(x,y) = x \cdot e^{xy} \cdot x \implies f_y(2,0) = 4$ ①

$Z = 2 + 1(x-2) + 4y$

$= 2 + x - 2 + 4y$

① $Z = x + 4y$

(b) Find the linearization of $f(x, y, z) = x^3\sqrt{y^2 + z^2}$ at the point $(1, 3, 4)$ and use it to estimate the number $(0.98)^3\sqrt{(3.01)^2 + (3.97)^2}$.

③ $\cdot\ L(x,y,z) = f(1,3,4) + f_x(1,3,4)(x-1) + f_y(1,3,4)(y-3) + f_z(1,3,4)(z-4)$

① $\cdot\ f(1,3,4) = 5$

① $\cdot\ f_x(x,y,z) = 3x^2\sqrt{y^2+z^2} \implies f_x(1,3,4) = 15$

① $\cdot\ f_y(x,y,z) = x^3 \dfrac{2y}{2\sqrt{y^2+z^2}} \implies f_y(1,3,4) = \dfrac{3}{5}$

① $\cdot\ f_z(x,y,z) = x^3 \dfrac{2z}{2\sqrt{y^2+z^2}} \implies f_z(x,y,z) = \dfrac{4}{5}$

$\cdot\ L(x,y,z) = 5 + 15(x-1) + \frac{3}{5}(y-3) + \frac{4}{5}(z-4)$ ①

or $= 15x + \frac{3}{5}y + \frac{4}{5}z - 15$

$\cdot\ (0.98)^3\sqrt{(3.01)^2+(3.97)^2} \approx 5 + 15(0.98-1) + \frac{3}{5}(3.01-3) + \frac{4}{5}(3.97-4)$ ③

$= 5 - 0.3 + 0.006 - 0.024$

$= 5 - 0.318$

$= 4.682$ ①

7. [12 **points**] Let $f(x,y) = x^2 + xy^3$. Find the unit vectors $\vec{u} =< u_1, u_2 >$ such that $D_{\vec{u}} f(1,-1) = 2$.

(2) • $\nabla f(x,y) = < f_x(x,y), f_y(x,y) > = < 2x + y^3, 3xy^2 >$

(1) $\nabla f(1,-1) = < 1, 3 >$

• We solve the system

$$\begin{cases} |\vec{u}| = 1 \\ D_{\vec{u}} f(1,-1) = 2 \end{cases} \Longrightarrow \begin{cases} \sqrt{u_1^2 + u_2^2} = 1 \\ \nabla f(1,-1) \cdot \vec{u} = 2 \end{cases}$$

$$\Longrightarrow \begin{cases} u_1^2 + u_2^2 = 1 \quad \text{---} (1) \\ u_1 + 3u_2 = 2 \quad \text{---} (2) \end{cases}$$ (3)

$$(2) \Longrightarrow u_1 = -3u_2 - 2 \overset{(1)}{\Longrightarrow} 9u_2^2 + 12u_2 + 4 + u_2^2 = 1$$

$$\Longrightarrow 10u_2^2 + 12u_2 + 3 = 0$$

$$\Longrightarrow u_2 = \frac{-12 \pm \sqrt{144 - 4(10)(3)}}{2(10)}$$

$$= \frac{-12 \pm \sqrt{24}}{20} = \frac{-6 \pm \sqrt{6}}{10}$$ (2)

• $u_2 = \dfrac{-6+\sqrt{6}}{10} \Longrightarrow u_1 = \dfrac{18 - 3\sqrt{6}}{10} - 2 = \dfrac{-2-3\sqrt{6}}{10}$

$$\Longrightarrow \vec{u} = \left\langle \frac{-2-3\sqrt{6}}{10}, \frac{-6+\sqrt{6}}{10} \right\rangle$$ (2)

• $u_2 = \dfrac{-6-\sqrt{6}}{10} \Longrightarrow u_1 = \dfrac{18+3\sqrt{6}}{10} - 2 = \dfrac{-2+3\sqrt{6}}{10}$

$$\Longrightarrow \vec{u} = \left\langle \frac{-2+3\sqrt{6}}{10}, \frac{-6-\sqrt{6}}{10} \right\rangle$$ (2)