3rd April → 11:30 → 17:00

Chap 4 Duality

Primal

$\min \; c^t x \qquad (3)$

s.t. $Ax = b$

$x \geq 0$

Dual

$\max \; \lambda^t b \qquad (4)$

s.t. $\lambda^t A \leq c^t$

$\lambda \geq 0$

Duality theorem

Lemma 1 If x and λ are feasible for (3) and (4), respectively, then $c^t x \geq \lambda^t b$.

If either of (3) or (4) has a finite optimal solution, so does the other, and the corresponding values of the objective functions are equal. If either problem has unbounded objective, the other problem has no feasible solution.

- For (3) suppose that $x = (x_B, 0)$ with basis B is a feasible basic optimal solution. We know that $x_B = B^{-1} b$ and reduced cost vector is $r_D^t = c_D^t - c_B^t B^{-1} D$

$(r_N^t = c_N^t - c_B^t B^{-1} N)$

Since $r_N^t \geq 0 \rightarrow c_B^t B^{-1} N \leq c_N^t$.

Let us suppose that $\lambda^t = c_B^t B^{-1} \rightarrow \lambda^t A = [\lambda^t B, \lambda^t N] = [c_B^t ; c_B^t B^{-1} N] \leq [c_B^t ; c_D^t] = c^t$

$\rightarrow \lambda^t A \leq c^t$, so λ is feasible for the dual.

and $\lambda^t b = c_B^t B^{-1} b = c_B^t x_B$ → this establised the optimality of λ for the dual.

Duality

Primal	Dual
obj max	obj min
$\leq$ constraint	Positive variable
$\geq$ "	negative variable
$=$ "	free variable
positive variable	$\leq$ constraint
negative variable	$\geq$ constraint
free variable	$=$ constraint

Ex

$$\max\ 2x + 3y + 5z \quad \text{s.t.} \quad \begin{cases} x + 2y + z \leq 3 \\ 3x + y + z \leq 4 \\ x + y + 2z \leq 5 \end{cases} \quad x, y, z \geq 0$$

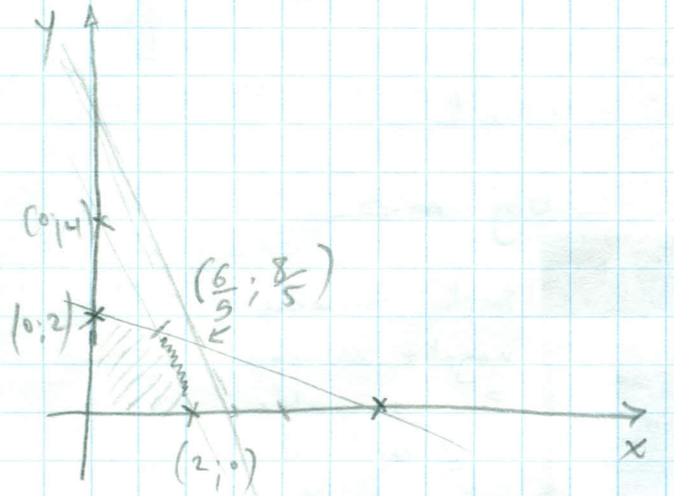
$$\Rightarrow$$

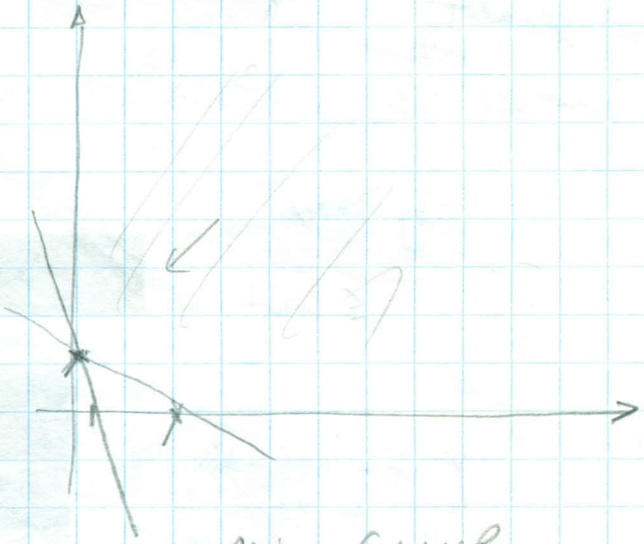
$$\min\ 3\alpha + 4\beta + 5\gamma \quad \text{s.t.} \quad \begin{cases} \alpha + 3\beta + \gamma \geq 2 \\ 2\alpha + \beta + \gamma \geq 3 \\ \alpha + \beta + 2\gamma \geq 5 \end{cases} \quad \alpha, \beta, \gamma \geq 0$$

Ex 1:

$$\max\ 2x + y \quad \text{s.t.} \quad \begin{cases} x + 3y \leq 6 \\ 2x + y \leq 4 \end{cases} \quad x; y \geq 0$$

$$\rightarrow \min\ 6\alpha + 4\beta \quad \text{s.t.} \quad \alpha + 2\beta \geq 2, \quad 3\alpha + \beta \geq 1, \quad \alpha; \beta \geq 0$$





max $2x+y$

s.t.

$$\begin{cases} x+3y \le 6 \to \alpha \\ 2x+y \le 4 \to \beta \end{cases}$$

$x, y \ge 0$

$Z(2;0)=4$

$Z(0;2)=2$

$Z\left(\frac{6}{5};\frac{8}{5}\right) = \frac{12+8}{5} = \frac{20}{5} = 4$

$x^* = \frac{6}{5}$; $y^* = \frac{8}{5} \to Z^* = 4$

min $6\alpha + 4\beta$

s.t.

$$\begin{cases} \alpha + 2\beta \ge 2 \\ 3\alpha + \beta \ge 1 \end{cases}$$

$\alpha ; \beta \ge 0$ $\qquad \alpha^* = 0 ; \beta^* = 1$

$Z^* = 4$

$x + 3y = 6 \to x = 6 - 3y$

$2x + y = 4 \qquad 12 - 6y + y = 4 \to 8 = 5y \Rightarrow y = 8/5$ and $x = 6 - \frac{24}{5} = \frac{6}{5}$

- what happens if the rhs of const 1 is reduced to 5 ?
- what happens if the rhs of const 2 is increased by 2 ?
- what happens if the obj. coef of y is increased ?
- what can you say about the dual optimal solution ?