4.4 Sensitivity & Complementary Slackness

• Complementary Slackness

① **theorem** (Assymetric form)

	Primal	Dual
	$\min c^t x$	$\max \lambda^t b$
	s.t $Ax = b$	s.t $\lambda^t A \leq c^t$
	$x \geq 0$	λ free

Let x and λ be feasible solutions for the primal and dual problems, a necessary and sufficient condition that they both be optimal solutions is that for all i:

(i) $x_i > 0 \Rightarrow \lambda^t a_i = c_i$

(ii) $x_i = 0 \Leftarrow \lambda^t a_i < c_i$

$\Rightarrow (\lambda^t A - c^t)x = 0 \to \lambda^t b = c^t x$

② **theorem** (Symetric form)

	Primal	Dual
	$\min c^t x$	$\max \lambda^t b$
	s.t. $Ax \geq b$	s.t $\lambda^t A \leq c^t$
	$x \geq 0$	$\lambda \geq 0$.

Let x and λ be feasible solution for the primal and dual problems, a necessary and sufficient condition that they both be optimal solution is that for all i and j:

(i) $x_i > 0 \Rightarrow \lambda^t a_i = c_i$

(ii) $x_i = 0 \Leftarrow \lambda^t a_i < c_i$

(iii) $\lambda_j > 0 \Rightarrow a^j x = b_j$

(iv) $\lambda_j = 0 \Leftarrow a^j x > b_j$

$\Rightarrow (\lambda^t A - c^t)x = 0$

$\Rightarrow \lambda^t (Ax - b) = 0$

4.5 Dual Simplex Method

- Feasibility for the dual but non-feasibility for the primal
- Starting a new optimization after changing the rhs b.

Given: $\min c^t x$

s.t $Ax = b$

$x \geq 0$

- Suppose B is a known basis such that $\lambda^t = c_B^t B^{-1}$ is feasible for the dual

$\Rightarrow \lambda^t a_j \leq c_j \quad \forall j = 1, 2, \dots, n$

Let: $\lambda^t a_j = c_j \quad \forall j = 1, \dots, m$

$\lambda^t a_j < c_j \quad \forall j = m+1, \dots, n$

we find a new vector $\bar{\lambda}$ such that one of the equalities becomes inequality and one of the inequalities becomes equality.

, while at the same time increasing the value of the dual objective function, the m equalities in the new solution would determine a new basis.

Bonus

Denote the i^{th} row of B^{-1} by u^i. then for $\bar{\lambda}^t = \lambda^t - \varepsilon u^i$

$$\Rightarrow \quad \bar{\lambda}^t a_j = \lambda^t a_j - \varepsilon u^i a_j$$

Let $z_j = \lambda^t a_j$ and $u^i a_j = y_{ij}$ (the ij^{th} element of the tableau)

$$\rightarrow \quad \bar{\lambda}^t a_j = c_j \quad ; \quad j = 1, 2, \ldots, m \; ; \; i \neq j$$

$$\bar{\lambda}^t a_i = c_i - \varepsilon$$

$$\bar{\lambda}^t a_j = z_j - \varepsilon y_{ij} \quad ; \quad j = m+1, m+2, \ldots, n$$

$$\Rightarrow \quad \bar{\lambda}^t b = \lambda^t b - \varepsilon x_{B_i}$$

$z_j - c_j$

$c_i - z_{ij}$

$c_j - z_j - \varepsilon y_{ij}$

- Select i such that the i^{th} component of x_B, $x_{B_i} < 0$
- if $y_{ij} < 0$ for some j, then $\varepsilon_0 = \min_j \left\{ \frac{z_j - c_j}{y_{ij}} \; ; \; y_{ij} < 0 \right\}$.

Ex: min $3x_1 + 4x_2 + 5x_2$

s.t.

$x_1 + 2x_2 + 3x_3 \geq 5$

$2x_1 + 2x_2 + x_3 \geq 6$

$x_1 ; x_2 ; x_3 \geq 0$

	x_1	x_2	x_3	e_1	e_2	
e_1	-1	-2	-3	1	0	-5
e_2	-2	-2	-1	0	1	-6
RC	3	4	5	0	0	

Since all reduced costs : $c_j - \overbrace{\lambda^t a_j}^{z_j}$ are non-negative $\rightarrow$ the basis is dual feasible.

($B = I = B^{-1}$; $\lambda^t = c_B^t B^{-1} = (0 \; 0)$; $RC = c_N^t - c_B B^{-1} N = (3 \; 4 \; 5)$)

- We select $x_{B_i} < 0 \rightarrow e_2 = -6$ to be removed from the basis
- To find the appropriate pivot : we select $\min_j \left\{ \frac{z_j - c_j}{y_{ij}} \; ; \; y_{ij} < 0 \right\}$.

$$\rightarrow \min \left\{ \frac{-3}{-2} ; \frac{-4}{-2} ; \frac{-5}{-1} \right\} = \frac{3}{2} \rightarrow j = 1 \quad x_1 \rightarrow \text{Enter the basis}$$

	3	4				
	x_1	x_2	x_3	e_1	e_2	
e_1	0	-1	$-\frac{5}{2}$	1	$-\frac{1}{2}$	-2
x_1	1	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	3
	0	1	$\frac{7}{2}$	0	$\frac{3}{2}$	

. Still dual feasible but not primal feasible

We select $e_1 = -2$

$$\min_j \left\{ \frac{z_j - c_j}{y_{ij}} \; ; \; y_{ij} < 0 \right\} = \min\left\{ \frac{-1}{-1} ; \frac{-\frac{7}{2}}{-\frac{5}{2}} ; \frac{-\frac{3}{2}}{-\frac{1}{2}} \right\}$$

$$= \min\left\{ 1 ; \frac{7}{5} ; 3 \right\} = 1$$

$\rightarrow j = 2$

x_2 enters

	x_1	x_2	x_3	e_1	e_2	
x_2	0	1	$\frac{5}{2}$	-1	$\frac{1}{2}$	2
x_1	0	0	-2	1	-1	1
	0	0	1	1	1	

$\Rightarrow$ Optimal for the primal.

Dual Simplex algorithm Ex 3.

$$\min Z = 3x_1 + 2x_2 + x_3$$

s.t.

$$\begin{cases} 3x_1 + x_2 + x_3 \geq 3 \\ -3x_1 + 3x_2 + x_3 \geq 6 \\ x_1 + x_2 + x_3 \leq 3 \end{cases}$$

$$x_1 ; x_2 ; x_3 \geq 0$$

$\Rightarrow$ $\min Z = 3x_1 + 2x_2 + x_3$

s.t.

$$\Rightarrow \begin{cases} -3x_1 - x_2 - x_3 + e_1 = -3 \\ +3x_1 - 3x_2 - x_3 + e_2 = -6 \\ x_1 + x_2 + x_3 + e_3 = 3 \end{cases}$$

$$x_1 ; x_2 ; x_3 \geq 0$$

	3	2	1				
	x_1	x_2	x_3	e_1	e_2	e_3	
e_1	-3	-1	-1	1	0	0	-3
e_2	3	-3	-1	0	1	0	-6
e_3	1	1	1	0	0	1	3
	3	2	1				

$\rightarrow$

$-4 \cdot -\frac{3}{2}$

e_2 leaves; $\min \left\{ \left| \frac{Rc_j}{a_{qj}} \right| ; a_{qj} < 0 \right\} = \min \left\{ \frac{2}{3} ; \frac{1}{1} \right\} = 2/3$

x_2 enters.

	3	2	1				
	x_1	x_2	x_3	e_1	e_2	e_3	
e_1	-4	0	$-2/3$	1	$-1/3$	0	-1
2 x_2	-1	1	$1/3$	0	$-1/3$	0	2
e_3	2	0	$2/3$	0	$1/3$	1	1
	5	0	$1/3$	0	$2/3$	0	

$\rightarrow$

	3	2	1				
	x_1	x_2	x_3	e_1	e_2	e_3	
1 x_3	6	0	1	$-3/2$	$\frac{1}{2}$	0	$3/2$
2 x_2	-3	1	0	$1/2$	$-1/2$	0	$3/2$
- e_3	-2	0	0	1	0	1	0
	3	0	0	$1/2$	$1/2$	0	

e_1 leaves; $\min \left\{ \frac{5}{4} ; \frac{1}{2} \right\} = \frac{1}{2} \Rightarrow x_3$ Enters.

Ex4.

min $3x_1+4x_2+5x_3$

$x_1+2x_1+3x_3 \geq 5$

$2x_1+2x_2+x_3 \geq 6$

		3	4	5			
		x_1	x_2	x_3	e_1	e_2	
0	e_1	-1	-2	-3	1	0	-5
0	e_2	-2	-2	-1	0	1	-6
		3	4	5	0	0	

Since all reduced costs are $\geq 0 \rightarrow$ the basis is primal unfeasible but dual feasible.

Select a variable leaving the basis and find the min

$$\min_j \left\{ \frac{z_j - c_j}{y_{ij}} \; ; \; y_{ij} < 0 \right\} = \min \left\{ \frac{-3}{-2} ; \frac{-4}{-2} ; \frac{-5}{-1} \right\} = \frac{3}{2}$$

$\Rightarrow x_1$ Enters.

		3	4	5	0	0	
		x_1	x_2	x_3	e_1	e_2	
0	e_1	0	-1	$-\frac{5}{2}$	1	$-\frac{1}{2}$	-2
+3	x_1	1	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	3
		0	1	$\frac{7}{2}$	0	$+\frac{3}{2}$	

$$\min \left\{ \frac{-1}{-1} ; \frac{-\frac{7}{2}}{-\frac{5}{2}} \right\} = 1$$

$5 + \frac{3}{2} = \frac{13}{2}$

$\frac{2 - \frac{3}{2}}{-\frac{1}{2}} \Rightarrow$

x_2 Enters:

		3	4	5			
		x_1	x_2	x_3	e_1	e_2	
4	x_2	0	1	$\frac{5}{2}$	-1	$\frac{1}{2}$	2
3	x_1	1	0	-2	1	-1	1
		0	0	1	1		

$5 - \left(4 \cdot \frac{5}{2} - 6\right) = 5 - (4) = 1$

$0 - (-4 + 3) = 1$

$0 - (2 - 3) = 1$