Final Exam for Math 571, 08 May 2018, Duration: 150 minutes

Note: In all questions, $f(.,.)$ is smooth and is assumed to satisfy a Lipschitz condition in the second variable. The approximate solution of $y^n := y(x_n)$ is denoted by y_n, where $y(x)$ is the solution of $y'(x) = f(x,y)$ on $(0,1)$ in the first six problems with $y(0) = y_0$ given. Let $x_n = nh$ for $0 \le n \le M$ with $h = 1/M$, and let $f_n := f(x_n, y_n)$, $f^n := f(x_n, y^n)$.

P. 1. Find α (> 0, real) such that the order of accuracy of the scheme below is as high as possible.

$$y_{n+2} - \alpha y_n = \frac{h}{3}[f_{n+2} + 4f_{n+1} + f_n].$$

Is the method convergent for this value of α?

P. 2. Show that the truncation error of the simplicit scheme below is of order h^2.

$$y_{n+1} - y_n = \frac{h}{2}(f_{n+1} + f_n), \quad n \ge 0.$$

P. 3. Consider the one-step method

$$y_{n+1} = y_n + \alpha h f(x_n, y_n) + \beta h f(x_n + \gamma h, y_n + \gamma h f(x_n, y_n)),$$

where α, β and γ are real parameters. Show that the method is consistent if and only if $\alpha + \beta = 1$. Show that the order of the method cannot exceed 2.

P. 4. Investigate the absolute stability using the Routh–Hurwitz criterion of the linear two-step method $y_{n+2} - y_n = \frac{h}{2}(f_{n+1} + 3f_n)$.

P. 5. Consider the following linear multistep method

$$y_{n+3} + (2\delta - 3)(y_{n+2} - y_{n+1}) - y_n = h\delta(f_{n+2} + f_{n+1}).$$

Find the values of the real parameter δ such that the linear multistep method is zero-stable.

P. 6. Find a and b such that the linear three-step method, $y_{n+1} + ay_{n-1} + by_{n-2} = hf_n$, is consistent.

P. 7. Consider the following two-point BVP:

$$-y''(x) + 2x(1-x)y'(x) + y(x) = 0 \quad \text{for } x \in (0,1), \quad \text{with } y'(0) = y'(1) = 0.$$

Define the standard continuous Galerkin piecewise linear finite element method.

P. 8. Develop a shooting method for the numerical solution of the problem below.

$$y''(x) = y(x)e^{y(x)} - 1 \quad \text{for } x \in (0,1), \quad \text{with } y(0) = y(1) = 0.$$

P. 9. Consider the following problem:

$$y''(x) + p(x)y'(x) + q(x)y(x) = f(x) \quad \text{for } x \in (0,1), \quad \text{with } y(0) + y'(0) = y(1) = 0,$$

where p and q are smooth. The finite difference solution is defined by: for $1 \le n \le M-1$,

$$\frac{y_{n+1} - 2y_n + y_{n-1}}{h^2} + p(x_n)\frac{y_{n+1} - y_{n-1}}{2h} + q(x_n)y_n = f(x_n),$$
$$y_0 + \frac{-3y_0 + 4y_1 - y_2}{2h} = 0, \quad y_M = 0\,.$$

Use Taylor series expansions to show that the truncation error $|T^n| \le Ch^2$ for $n \ge 1$.