King Fahd University of Petroleum & Minerals
Department of Mathematics and Statistics

MATH 568, Semester 172 (2017-2018)

Final EXAM
May 9, 2018

Allowed Time: 3 hours.

Student Name:

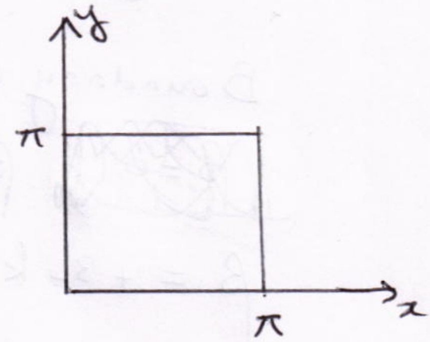
Student ID Number:

Instructions:
1. Write neatly and legibly -- *you may lose points for messy work.*
2. Show all your work -- *no points for answers without justification.*
3. Make sure that you have 5 problems.

Problem No.	Points	Maximum Points
1		16
2		6
3		16
4		16
5		16
Total:		**70**

Problem 1. Solve the problem

$$\begin{cases} \Delta u(x,y) + 3u(x,y) = 0, & in\ (0,\pi)\times(0,\pi) \\ u(\pi,y) = -2sin5y \\ u(x,0) = u(x,\pi) = u(0,y) = 0 \end{cases}$$



Sol. We use separation of variables

let $u(x,y) = X(x)\,Y(y)$

So, the equation become

$$XY'' + X''Y = -3XY$$

$$\Rightarrow \frac{X''}{X} + \frac{Y''}{Y} = -3 \Leftrightarrow \frac{Y''}{Y} = -3 - \frac{X''}{X} = -\lambda.$$

Boundary conditions $\Rightarrow Y(0) = Y(\pi) = 0$

Thus, we solve (1)

$$Y'' + \lambda Y = 0, \quad Y(0) = Y(\pi) = 0$$

$\boxed{\lambda = 0} \Rightarrow Y = ay + b.$ BC's $\Rightarrow Y \equiv 0$ (reject).

$\boxed{\lambda = -k^2} \Rightarrow$

$$Y = c_1 e^{ky} + c_2 e^{-ky}$$

$Y(0) = 0 \Leftrightarrow c_1 + c_2 = 0 \Rightarrow Y = \hat{c}\sinh yk$

$Y(\pi) = 0 \Leftrightarrow \hat{c}\sinh k\pi = 0 \Leftrightarrow \hat{c} = 0 \Rightarrow Y \equiv 0.$

Also, this is not acceptable.

$\boxed{\lambda = k^2 > 0} \Rightarrow Y = c_1 \cos ky + c_2 \sin ky$

$Y(0) = 0 \Rightarrow c_1 = 0 \Rightarrow Y = c_2 \sin ky$

$Y(\pi) = c_2 \sin k\pi = 0 \Rightarrow k = 1, 2, 3, \dots$

So, $Y_k(y) = \sin ky$ solves ①, for $\lambda = k^2 > 0$,

$k = 1, 2, \dots$

Next, $\dfrac{X''}{X} = \lambda - 3 \Rightarrow X'' + (3-\lambda)X = 0$

or $X'' + (3-k^2)X = 0 \Leftrightarrow X'' + \beta X = 0 \; (\beta = 3-k^2)$

Boundary conditions: $X(0) = 0$

$\beta = 0$ is out off question.

$\beta = +3-k^2 > 0$, $k = 1$ only $\Rightarrow X'' + 2X = 0$

$\Rightarrow X = c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x$

BC's $\Rightarrow X(0) = 0 \Rightarrow c_1 = 0 \Rightarrow X_1 = \sin\sqrt{2}x$ is

a solution of $X'' + 2X = 0$, $X(0) = 0$.

Next, for $\beta = 3-k^2 < 0$, $k = 2, 3, \dots$, we have

$$X = c_1 e^{\sqrt{k^2-3}\,x} + c_2 e^{-\sqrt{k^2-3}\,x}$$

$X(0) = 0 \Leftrightarrow c_1 + c_2 = 0 \Rightarrow X_k = \sinh(\sqrt{k^2-3})x$

is a solution of $X'' - (k^2-3)X = 0$, $X(0) = 0$, $k \geq 2$.

Thus $u_1(x,y) = \sin\sqrt{2}x \sin y$.

$u_k = \sinh(\sqrt{k^2-3})x \sin ky$, $k = 2, 3, \dots$

Thus

$$u(x,y) = a_1 \sin\sqrt{2}x \sin y + \sum_{k=2}^{\infty} a_k \sinh(\sqrt{k^2-3})x \sin ky$$

is a solution of $\begin{cases} \Delta u + 3u = 0 \\ u(0,y) = u(x,0) = u(x,\pi) = 0 \end{cases}$

$$u(\pi, y) = a_1 \sin\sqrt{2}\pi \sin y$$

$$+ \sum_{k=2}^{\infty} a_k \sinh(\sqrt{k^2-3})\pi \sin ky = -2\sin 5y$$

$$\Rightarrow a_1 = 0, \quad a_k = 0, \ \forall k \neq 5, \quad a_5 = \frac{-2}{\sinh\sqrt{22}\,\pi}.$$

Thus the solution is

$$\boxed{u(x,y) = \frac{-2}{\sinh\sqrt{22}\,\pi} \sin\sqrt{22}x \sin 5y}$$

Problem 2. Check if the following problem has a solution

$$\begin{cases} -\Delta u = xy, & in\ D \\ \dfrac{\partial u}{\partial \eta} = 2, & on\ \partial D, \end{cases}$$

where $D = \{(x, y)\ /\ \ x^2 + y^2 < 1\}$.

Sol. We have to check, first if $\iint_D -xy = \int_{\partial D} 2$

- $\iint_D -xy = \int_0^{2\pi}\int_0^1 -(r\cos\theta)(r\sin\theta)\, r\, dr\, d\theta$

$$= -\int_0^{2\pi} \cancel{\sin\theta\cos\theta\, d\theta}^{\,0} \cdot \int_0^1 r^3\, dr = 0.$$

- $\int_{\partial D} 2\, d\theta = 2\int_0^{2\pi} d\theta = 4\pi$

Since $\iint -xy \neq \int_{\partial D} 2$. Then problem has no solution

Problem 3. Solve the problem

$$\begin{cases} u_t(x,y,t) - \Delta u(x,y,t) = u(x,y,t), & in\ \mathbb{R}^2 \times (0,\infty) \\ u(x,y,0) = x + y \end{cases}$$

Sol. let $u = v e^{\alpha t}$, for α to be chosen

$$u_t = v_t e^{\alpha t} + \alpha v e^{\alpha t}, \quad \Delta u = \Delta v\, e^{\alpha t}$$

Substitute in the equation:

$$(v_t + \alpha v - \Delta v) e^{\alpha t} = v e^{\alpha t}$$

For $\alpha = 1$, we get $\begin{cases} v_t - \Delta v = 0 \\ v(x,y,0) = x + y \end{cases}$

The solution is

$$v(x,y,t) = \frac{1}{4\pi t} \iint_{\mathbb{R}^2} e^{-\frac{|(x,y)-(\xi,\eta)|^2}{4t}} (\xi + \eta)\, d\xi\, d\eta$$

$$= \frac{1}{4\pi t} \iint_{\mathbb{R}^2} e^{-\frac{(x-\xi)^2}{4t}} \cdot e^{-\frac{(y-\eta)^2}{4t}} (\xi + \eta)\, d\xi\, d\eta$$

$$= \frac{1}{4\pi t} \int_{-\infty}^{+\infty} e^{-\frac{(y-\eta)^2}{4t}} d\eta \int_{-\infty}^{+\infty} e^{-\frac{(x-\xi)^2}{4t}} \xi\, d\xi$$

$$+ \frac{1}{4\pi t} \int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4t}} d\xi \int_{-\infty}^{+\infty} e^{-\frac{(y-\eta)^2}{4t}} \eta\, d\eta$$

$$\cdot \int_{-\infty}^{\infty} e^{-\frac{(y-\eta)^2}{4t}}\, d\eta = \int_{-\infty}^{\infty} e^{-z^2} \sqrt{4t}\, dz = \sqrt{4\pi t}\,, \quad z = \frac{y-\eta}{\sqrt{4t}}$$

Similarly,

we have $\displaystyle\int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4t}}\, d\xi = \sqrt{4\pi t}\,.$

Next,

$$\cdot \int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4t}}\, \xi\, d\xi = \int_{-\infty}^{\infty} e^{-z^2} (x-z)\sqrt{4t}\, dz$$

$$= x\sqrt{4t} \int_{-\infty}^{\infty} e^{-z^2}\, dz - \cancelto{0}{\int_{-\infty}^{\infty} e^{-z^2} z\, dz}$$

$$= \sqrt{4\pi t}\; x\,.$$

Similarly,

$$\int_{-\infty}^{\infty} e^{-\frac{(y-\eta)^2}{4t}}\, \eta\, d\eta = \sqrt{4\pi t}\; y\,.$$

So, $v(x,y,t) = \frac{1}{\cancel{4\pi t}} \cdot \cancel{4\pi t}\,(x+y) = x+y.$

$$\Rightarrow \boxed{u(x,y,t) = (x+y)\, e^{t}}$$

Problem 4. Write down a solution for the problem

$$\begin{cases} u_{tt}(x,t) - 4u_{xx}(x,t) = f(x,t), & x>0, t>0 \\ u(0,t) = 0, & t \geq 0 \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & x>0 \end{cases}$$

where $u_0(0) = u_1(0) = f(0,t) = 0, \ \forall t \geq 0$.

Sol. We extend the functions u_0, u_1, f on $\mathbb{R}$ as odd functions. So

$$\tilde{u}_0(x) = \begin{cases} u_0(x), & x \geq 0 \\ -u_0(-x), & x<0 \end{cases}, \quad \tilde{u}_1(x) = \begin{cases} u_1(x), & x \geq 0 \\ -u_1(-x), & x<0 \end{cases}$$

$$\tilde{f}(x,t) = \begin{cases} f(x,t), & x \geq 0 \\ -f(-x,t), & x<0 \end{cases}$$

We then consider the Cauchy nonhomogeneous pb:

$$\begin{cases} \tilde{u}_{tt} - 4\tilde{u}_{xx} = \tilde{f}, & x \in \mathbb{R}, t>0 \\ \tilde{u}(0,t) = 0, & t \geq 0 \\ \tilde{u}(x,0) = \tilde{u}_0(x), \quad \tilde{u}_t(x,0) = \tilde{u}_1(x), & x \in \mathbb{R}. \end{cases}$$

The solution is

$$\tilde{u}(x,t) = \frac{1}{2}\left[\tilde{u}_0(x+2t) + \tilde{u}_0(x-2t)\right] + \frac{1}{4}\int_{x-2t}^{x+2t} \tilde{u}_1(s)\,ds + \frac{1}{4}\int_0^t \int_{x-2(t-s)}^{x+2(t-s)} \tilde{f}(\eta,s)\,d\eta\,ds.$$

in the region $\underline{x \geq 2t}$, we have

$$u(x,t) = \frac{1}{2}\left[u_0(x+2t) + u_0(x-2t)\right] + \frac{1}{4}\int_{x-2t}^{x+2t} u_1(s)\,ds$$

$$+ \frac{1}{4}\int_0^t \int_{x-2(t-s)}^{x+2(t-s)} f(\eta,s)\,d\eta\,ds.$$

In the region $\underline{x < 2t}$, we have

$$\tilde{u}_0(x-2t) = -u_0(2t-x)$$

$$\int_{x-2t}^{x+2t} \tilde{u}_1(s)\,ds = \int_{x-2t}^{0} -u_1(-s)\,ds + \int_0^{x+2t} u_1(s)\,ds = \int_{2t-x}^{0} u_1(\xi)\,d\xi + \int_0^{x+2t} u_1(s)\,ds$$

$$= \int_{2t-x}^{2t+x} u_1(s)\,ds.$$

$$\int_0^t \int_{x-2(t-s)}^{x+2(t-s)} \tilde{f}(\eta,s)\,d\eta\,ds = \int_0^{t-\frac{x}{2}} \int_{x-2(t-s)}^{x+2(t-s)} \tilde{f}(\eta,s)\,d\eta\,ds + \int_{t-\frac{x}{2}}^{0} \int_{x-2(t-s)}^{x+2(t-s)} \tilde{f}(\eta,s)\,d\eta\,ds$$

$$= \int_{t-\frac{x}{2}}^{t} \int_{x-2(t-s)}^{x+2(t-s)} f(\eta,s)\,d\eta\,ds + \int_0^{t-\frac{x}{2}} \int_{x-2(t-s)}^{0} -f(-\eta,s)\,d\eta\,ds$$

$$+ \int_0^{t-\frac{x}{2}} \int_0^{x+2(t-s)} f(\eta,s)\,d\eta\,ds$$

$$= \int_{t-\frac{x}{2}}^{t} \int_{x-2(t-s)}^{x+2(t-s)} f(\eta,s)\,d\eta\,ds + \int_{0}^{t-\frac{x}{2}} \int_{2(t-s)-x}^{0} f(\xi,s)\,d\xi\,ds$$

$$+ \int_{0}^{t-\frac{x}{2}} \int_{0}^{x+2(t-s)} f(\eta,s)\,d\eta\,ds$$

$$= \int_{0}^{t-\frac{x}{2}} \int_{2(t-s)-x}^{2(t-s)+x} f(\eta,s)\,d\eta\,ds + \int_{t-\frac{x}{2}}^{t} \int_{x-2(t-s)}^{x+2(t-s)} f(\eta,s)\,d\eta\,ds.$$

So, the solution is

$$u(x,t) = \begin{cases} \frac{1}{2}\left[u_0(x+2t)+u_0(x-2t)\right] + \frac{1}{4}\int_{x-2t}^{x+2t} u_1(s)\,ds + \frac{1}{4}\int_{0}^{t}\int_{x-2(t-s)}^{x+2(t-s)} f(\eta,s)\,d\eta\,ds, & x \geq 2t \\ \frac{1}{2}\left[u_0(2t+x)-u_0(2t-x)\right] + \frac{1}{4}\int_{2t-x}^{2t+x} u_1(s)\,ds \\ \quad + \frac{1}{4}\int_{0}^{t-\frac{x}{2}} \int_{2(t-s)-x}^{2(t-s)+x} f(\eta,s)\,d\eta\,ds + \frac{1}{4}\int_{t-\frac{x}{2}}^{t} \int_{x-2(t-s)}^{x+2(t-s)} f(\eta,s)\,d\eta\,ds, & x < 2t \end{cases}$$

Problem 5. Solve the problem

$$\begin{cases} u_{tt}(x,y,z,t) - 9\Delta u(x,y,z,t) = 0, & in\ \mathbb{R}^3 \times (0,\infty) \\ u(x,y,z,0) = z + xy, & in\ \mathbb{R}^3 \\ u_t(x,y,z,0) = 1 + x^2, & in\ \mathbb{R}^3 \end{cases}$$

Sol. Let $X = (x, y, z)$. So

$$u(X,t) = \frac{\partial}{\partial t}\left[\frac{t}{4\pi}\int_{|\xi|=1} u_0(X+t\xi)\, dS_\xi\right] + \frac{t}{4\pi}\int_{|\xi|=1} u_1(X+t\xi)\, dS_\xi$$

$$= \frac{\partial}{\partial t}\left(\frac{t}{4\pi}\int_{|\xi|=1}\left[(z+3t\xi_3)+(x+3t\xi_1)(y+3t\xi_2)\right] dS_\xi\right]$$

$$+ \frac{t}{4\pi}\int_{|\xi|=1}\left[1+(x+3t\xi_1)^2\right] dS_\xi .$$

$$I_1 = \int_{|\xi|=1}(z+3t\xi_3)\, dS_\xi + \int_{|\xi|=1} xy\, dS_\xi + 3tx\int_{|\xi|=1}\xi_2\, dS_\xi + 3ty\int_{|\xi|=1}\xi_1\, dS_\xi$$

$$+ 9t^2\int_{|\xi|=1}\xi_1\xi_2\, dS_\xi$$

$= 4\pi(z+xy) +$ other integrals.

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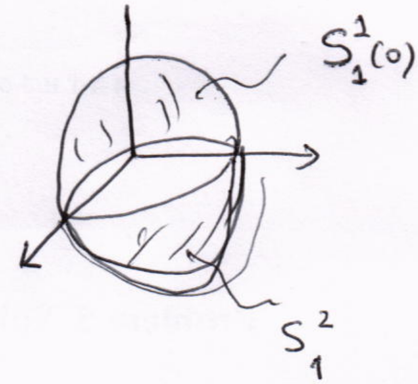
$$\int_{S_1(0)} \xi_3 \, dS_\xi = \int_{S_1^1(0)} \xi_3 \, dS_\xi + \int_{S_2^2(0)} \xi_3 \, dS_\xi$$



on $S_1^1(0)$, $\xi_3 = \sqrt{1-\xi_1^2-\xi_2^2} = \psi(\xi_1,\xi_2)$

So $$\int_{S_1^1(0)} \xi_3 \, dS_\xi = \iint_D \sqrt{1-\xi_1^2-\xi^2} \sqrt{1+\psi_{\xi_1}^2+\psi_{\xi_2}^2} \, d\xi_1 d\xi_2$$

on $S_1^2(0)$; $\xi_3 = -\psi(\xi_1,\xi_2)$

So $$\int_{S_1^2(0)} \xi_3 \, dS_\xi = -\iint_D \sqrt{1-\xi_1^2-\xi_2^2} \sqrt{1+\psi_{\xi_1}^2+\psi_{\xi_2}^2} \, d\xi_1 d\xi_2$$

$$\Rightarrow \int_{S_1(0)} \xi_3 \, dS_\xi = 0 \quad \blacksquare$$

$$\int_{S_1(0)} \xi_1 \, dS_\xi = 2\iint_D \xi_1 \sqrt{1+\psi_{\xi_1}^2+\psi_{\xi_2}^2} \, d\xi_1 d\xi_2$$

$$\Rightarrow 1+\psi_{\xi_1}^2+\psi_{\xi_2}^2 = \frac{1}{1-\xi_1^2-\xi_2^2}$$

Thus $$\int_{S_1(0)} \xi_1 \, dS_\xi = 2\int_0^{2\pi}\int_0^1 \frac{r\cos\theta}{\sqrt{1-r^2}} r\,dr\,d\theta = 0$$

Similarly, we have

$$\int_{S_1(0)} \xi_2 \, dS_\xi = 2\int_0^{2\pi}\int_0^1 \frac{r\sin\theta}{\sqrt{1-r^2}}\, r\,dr\,d\theta = 0$$

Also, $$\iint_{S_1(0)} \xi_1\xi_2 \, dS_\xi = \int_0^{2\pi}\int_0^1 \frac{r^2\cos\theta\sin\theta}{\sqrt{1-r^2}}\, r\,dr\,d\theta = 0 .$$

Thus $$\boxed{I_1 = 4\pi(z+xy)}$$

Now,

$$I_2 = \int_{|\xi|=1} \left[1 + (x+3t\xi_1)^2\right] dS_\xi = \int_{|\xi|=1} dS_\xi + x^2 \int_{|\xi|=1} dS_\xi$$

$$+ \cancel{6x\int_{|\xi|=1} \xi_1 \, dS_\xi}^{\,0} + 9t^2 \int_{|\xi|=1} \xi_1^2 \, dS_\xi$$

$$= 4\pi(1+x^2) + 9t^2\left(2\int_0^{2\pi}\int_0^1 \frac{r^2\cos^2\theta}{\sqrt{1-r^2}}\, r\,dr\,d\theta\right)$$

$$* \int_0^{2\pi}\int_0^1 \frac{r^2\cos^2\theta}{\sqrt{1-r^2}}\, r\,dr\,d\theta = \int_0^1 \frac{r^3}{\sqrt{1-r^2}}\,dr \int_0^{2\pi} \frac{1+\cos 2\theta}{2}\,d\theta$$

$$= \pi \int_0^1 \frac{r^3}{\sqrt{1-r^2}}\,dr$$

let $1-r^2 = s \Rightarrow r^2 = 1-s, \quad ds = -2r\,dr$

So $$\int_0^1 \frac{r^3\,dr}{\sqrt{1-r^2}} = \frac{-1}{2}\int_1^0 \frac{1-s}{\sqrt{s}}\,ds$$

$$= +\frac{1}{2}\int_0^1 \left(s^{-\frac{1}{2}} - s^{\frac{1}{2}}\right)ds = \frac{1}{\cancel{2}}\left[\cancel{2}s^{\frac{1}{2}} - \frac{\cancel{2}}{3}s^{\frac{3}{2}}\right]_0^1$$

$$= \frac{2}{3}$$

Thus $$I_2 = 4\pi(1+x^2) + 18t^2\,\frac{2\pi}{3}$$

So, the solution is

$$u(X,t) = \frac{\partial}{\partial t}\left(\frac{t\,\cancel{4\pi}}{\cancel{4\pi}}(3+xy)\right) + \frac{t}{4\pi}\left[4\pi(1+x^2) + \frac{12}{3}\pi t^2\right]$$

$$\boxed{u(X,t) = 3 + xy + t(1+x^2) + 3t^3}$$