King Fahd University of Petroleum & Minerals
Department of Mathematics and Statistics

MATH 568, Semester 172 (2017-2018)

EXAM I

March 15, 2018

Allowed Time: 150 mns.

Student Name:

Student ID Number:

Instructions:

1. Write neatly and legibly -- *you may lose points for messy work.*
2. Show all your work -- *no points for answers without justification.*
3. Make sure that you have 4 problems.

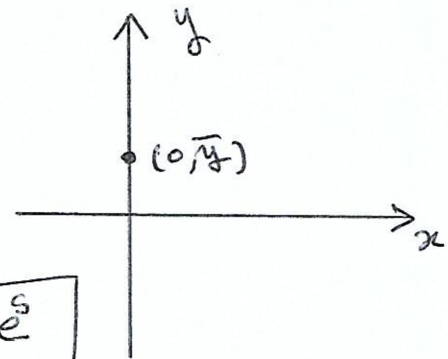
Problem No.	Points	Maximum Points
1		10
2		25
3		10
4		25
Total:		**70**

Problem 1. Solve the problem

$$u_x(x,y) + y u_y(x,y) = u^2(x,y), \qquad u(0,y) = \frac{1}{y}, \qquad y \neq 0$$

Solution

Characteristic system:



$$\frac{dx}{ds} = 1, \quad x(0) = 0 \Rightarrow \boxed{x = s}$$

$$\frac{dy}{ds} = y, \quad y(0) = \bar{y} \Rightarrow \boxed{y = \bar{y} e^{s}}$$

$$\frac{du}{ds} = u^2, \qquad u(0) = \frac{1}{\bar{y}}$$

So, we have

$$\frac{du}{u^2} = ds \Rightarrow -\frac{1}{u} = s + c \Rightarrow u = \frac{-1}{s+c}$$

$$\Rightarrow u(0) = -\frac{1}{c} = \frac{1}{\bar{y}} \Rightarrow c = -\bar{y}$$

$$\Rightarrow u = \frac{-1}{s - \bar{y}}$$

Thus,

$$\boxed{u(x,y) = \frac{-1}{x - y e^{-x}}}.$$

Problem 2. Find a solution to the problem

$$u_x(x,y)u_y(x,y) = -u^2(x,y), \qquad u(x,x) = 1$$

Is the solution unique?

Solution The equation can be written as

$F(x, y, p, q, u) = pq + u^2 = 0$, such that $p = u_x, q = u_y$

characteristic system

$\frac{dx}{ds} = F_p = q, \quad x(0) = \bar{x}$

$\frac{dy}{ds} = F_q = p, \quad y(0) = \bar{x}$

$\frac{du}{ds} = pF_p + qF_q = 2pq = -2u^2, \quad u(0) = 1.$

$\frac{dp}{ds} = -F_x - pF_u = -2pu, \quad p(0) = \phi(\bar{x})$

$\frac{dq}{ds} = -F_y - qF_u = -2qu, \quad q(0) = \psi(\bar{x})$

From $u(x,x) = 1$, we have

$u_x + u_y = 0 \Rightarrow p(0) + q(0) = 0$

The equation $\Rightarrow p(0)\, q(0) = -u^2(0) = -1$

Thus, we get

$\phi(\bar{x}) = 1, \ \psi(\bar{x}) = -1$ OR $\phi(\bar{x}) = -1, \ \psi(\bar{x}) = 1$

$$\frac{du}{ds} = -2u^2, \; u(0) = 1 \Rightarrow -\frac{1}{u} = -2s + c$$

using $u(0) = 1$, we obtain

$$\boxed{u = \frac{1}{2s+1}}$$

$$\frac{dp}{ds} = -2up = \frac{-2}{2s+1}p, \quad p(0) = 1$$

$$\Rightarrow \frac{dp}{p} = -\frac{2\,ds}{2s+1} \Rightarrow \ln|p| = -\ln|2s+1| + c$$

$$\Rightarrow p = \frac{c'}{2s+1} \Rightarrow p(0) = c' = 1$$

$$\Rightarrow \boxed{p = \frac{1}{2s+1}}$$

$$\frac{dq}{ds} = -2qu = -\frac{2}{2s+1}q, \quad q(0) = -1$$

$$\Rightarrow \boxed{q = \frac{-1}{2s+1}}$$

$$\frac{dx}{ds} = q = \frac{-1}{2s+1}, \quad x(0) = \bar{x}$$

$$\Rightarrow x = -\frac{1}{2}\ln|2s+1| + c$$

$$x(0) = c = \bar{x} \Rightarrow \boxed{x = -\frac{1}{2}\ln|2s+1| + \bar{x}.}$$

$$\frac{dy}{ds} = p = \frac{1}{2s+1}, \quad y(0) = \bar{x}$$

$$\Rightarrow \boxed{y = \frac{1}{2}\ln|2s+1| + \bar{x}}$$

Now,

$$y - x = \ln|2s+1| = \ln|2s+1|$$

$$\Rightarrow \quad e^{y-x} = |2s+1|$$

$$\Rightarrow \quad 2s+1 = \pm e^{y-x}$$

$$\Rightarrow \quad u = \pm e^{x-y}$$

But $u(x,x) = 1 \Rightarrow \boxed{u(x,y) = e^{x-y}}$

The solution is not unique since

$$u(x,y) = e^{y-x}$$

is also a solution

Notice that if we take $p(0) = -1$ and $q(0) = 1$,

we get $u(x,y) = e^{y-x}$.

Problem 3. Find an appropriate coordinate system, in which

$$u_{xx}(x,y) + y^2 u_{yy}(x,y) = 0, \qquad y \neq 0$$

Takes the canonical form. **Do not solve**

<u>Solution</u> $\delta = -y^2 < 0 \Rightarrow$ pde is elliptic

$$\frac{dy}{dx} = \pm i y \Leftrightarrow \frac{dy}{y} = \pm i\, dx \Rightarrow \ln|y| = \pm i x + c$$

So, we can take $\phi = \ln|y| + ix$

Consequently, $\xi = \ln|y|$ and $\eta = x$.

If $u(x,y) = W(\xi,\eta)$ then

$$u_x = W_\eta, \qquad u_{xx} = W_{\eta\eta}$$

$$u_y = \frac{1}{y} W_\eta \qquad u_{yy} = -\frac{1}{y^2} W_\xi + \frac{1}{y^2} W_{\xi\xi}$$

Substitute in pde:

$$W_{\eta\eta} + y^2\left(-\frac{1}{y^2} W_\xi + \frac{1}{y^2} W_{\xi\xi}\right) = 0$$

Thus, the equation is

$$\boxed{W_{\xi\xi} + W_{\eta\eta} = W_\xi}$$

Problem 4. Given the linear second-order equation:

$$u_{xx} - 2xyu_{xy} + x^2y^2u_{yy} = y(1-x^2)u_y + 6x \qquad (*)$$

a. Show, by using an appropriate coordinate system, that (*) can be reduced to

$$w_{\eta\eta} = 6\eta \qquad (**)$$

b. Find the general solution of (**)

c. Find the solution, if $u(0,y) = u_x(0,y) = y$

Solution

(a) $\delta = b^2 - ac = x^2y^2 - x^2y^2 = 0 \Rightarrow$ pde is parabolic

Characteristic:

$$\frac{dy}{dx} = -xy \Rightarrow \frac{dy}{y} = -x\,dx \Rightarrow \ln|y| = -\frac{x^2}{2} + c$$

$$\Rightarrow |y| = e^{-\frac{x^2}{2}} \cdot e^{c} \Rightarrow y = k\,e^{-\frac{x^2}{2}}$$

$$\Rightarrow y e^{\frac{x^2}{2}} = k = \xi$$

take $\eta = x$

Then $J = \begin{vmatrix} yx\,e^{\frac{x^2}{2}} & 1 \\ e^{\frac{x^2}{2}} & 0 \end{vmatrix} = -e^{\frac{x^2}{2}} \neq 0$

Thus (ξ, η) is a coordinate system.

let $u(x,y) = W(\xi,\eta)$. So,

$$u_x = xy e^{\frac{x^2}{2}} W_\xi + W_\eta$$

$$u_{xx} = x^2y^2 e^{x^2} W_{\xi\xi} + 2xy e^{\frac{x^2}{2}} W_{\xi\eta} + W_{\eta\eta} + e^{\frac{x^2}{2}}(y + x^2y) W_\xi$$

$$u_y = e^{\frac{x^2}{2}} W_\xi, \qquad u_{yy} = e^{x^2} W_{\xi\xi}$$

$$u_{xy} = xy e^{x^2} W_{\xi\xi} + e^{\frac{x^2}{2}} W_{\xi\eta} + x e^{\frac{x^2}{2}} W_\xi$$

Substitute in the pde:

$$(\cancel{x^2y^2 - 2x^2y^2 + x^2y^2}) e^{x^2} W_{\xi\xi} + (\cancel{2xy - 2xy}) e^{\frac{x^2}{2}} W_{\xi\eta}$$

$$+ W_{\eta\eta} + e^{\frac{x^2}{2}}(y + x^2y - 2x^2y) W_\xi = y(1-x^2)u_y + 6x$$

$$W_{\eta\eta} + \cancel{e^{\frac{x^2}{2}} y(1-x^2) e^{-\frac{x^2}{2}} u_y} = \cancel{y(1-x^2)u_y} + 6x$$

$$\boxed{W_{\eta\eta} = 6\eta}$$

(b) $W_\eta = 3\eta^2 + f(\xi)$

$$W = \eta^3 + \eta f(\xi) + g(\xi)$$

So $u(x,y) = x^3 + x f(y e^{\frac{x^2}{2}}) + g(y e^{\frac{x^2}{2}})$,

$f, g \in C^2$

(c) $u(0,y) = g(y) = y$

$u_x(x,y) = 3x^2 + f(ye^{\frac{x^2}{2}}) + x^2 f'(ye^{\frac{x^2}{2}}) + x g'(ye^{\frac{x^2}{2}})$

$u_x(0,y) = f(y) = y$

Thus

$$\boxed{u(x,y) = x^3 + xye^{\frac{x^2}{2}} + ye^{\frac{x^2}{2}}}$$