MATH 533 (COMPLEX ANALYSIS) MIDTERM EXAM

Student ID:

Name:

In this exam, G always means a connected open region in $\mathbb{C}$, if there is no comment about it.

1. A C^1-function $f : G \to \mathbb{C}$ is said to be *anti-holomorphic* if the function $g(z) := f(\bar{z})$ is holomorphic. Prove that a funtion which is both holomorphic and anti-holomorphic is a constant.

2. Let $f : G \setminus \{a\} \to \mathbb{C}$ be an analytic funtion for some $a \in G$. Suppose there exist two sequences $\{a_n\}$ and $\{b_n\}$ in $G \setminus \{a\}$ such that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = a, \quad \lim_{n\to\infty} f(a_n) = 0, \quad \lim_{n\to\infty} f(b_n) = 1.$$

Then show that there exists a sequence $\{c_n\}$ in $G \setminus \{a\}$ such that

$$\lim_{n\to\infty} c_n = a, \quad \lim_{n\to\infty} f(c_n) = 2.$$

3. Determine whether or not the function

$$f(z) = \frac{5z - 2}{z(z-1)(z+2)}$$

has a primitive on $\{z \in \mathbb{C} : |z| > 3\}$.

4. Let $B = B(0;1) = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc and $C = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle. Let f be a function which is analytic on a neighborhood of $\overline{B} = \{z \in \mathbb{C} : |z| \leq 1\}$.

(a) Prove that for any $a \in B$ and for any positive integer n

$$(f(a))^n = \frac{1}{2\pi i} \int_\gamma \frac{(f(z))^n}{z - a}\, dz,$$

where $\gamma = e^{it}$, $0 \leq t \leq 2\pi$ is the standard parametrization of C.

(b) If we assume $|f(z)| \leq M$ for all $z \in C$ and for some constant $M > 0$, then show that

$$|f(a)|^n \leq M^n/d$$

for all positive integer n, where d represents the distance from a to C.

(c) Without using Maximum Modulus Theorem, show that

$$\max_{z \in \overline{B}} |f(z)| = \max_{z \in C} |f(z)|$$

from (b).

5. Let $f : G \to \mathbb{C}$ be an analytic function with zeros $a_1, \dots, a_n$ in G (repeated according to multiplicities). Suppose γ is a piecewise smooth closed curve in G such that $\gamma \approx 0$ in G which does not pass through $a_1, \dots, a_n$. Show that

$$\frac{1}{2\pi i} \int_\gamma \frac{z f'(z)}{f(z)}\, dz = \sum_{j=1}^{n} a_j\, n(\gamma; a_j).$$

(Hint. Recall the proof of the Argument Principle.)

6. (a) Let $f : G \to \mathbb{C}$ be an analytic function. Then for any real-valued harmonic function ϕ defined on $f(G)$, show that $\psi := \phi \circ f$ is harmonic on G.

(b) Show $\phi(z) = \operatorname{Re} z = x$ is a harmonic function on the strip $\{z = x + iy \in \mathbb{C} : 0 < x < 1\}$ such that $\phi \equiv 0$ on the line $L_1 = \{x = 0\}$ and $\phi \equiv 1$ on $L_2 = \{x = 1\}$.

(c) Let $C_1 := \{z : |z - 2| < 2\}$ and $C_2 := \{z : |z - 1| < 1\}$. Find a real-valued harmonic function ψ on G =the region between C_1 and C_2 such that $\psi \equiv 0$ on $C_1 \setminus \{0\}$ and $\psi \equiv 1$ on $C_2 \setminus \{0\}$.