King Fahd University of Petroleum and Minerals
Department of Mathematics & Statistics
Math 531 (Real Analysis) Major Exam II Spring 2018(172)- 120 minutes

ID#:________________ NAME:______________________________

Notation: $\mathbb{R}$ = the real numbers, $\mathbb{N}$ = the natural numbers, m = Lebesgue measure.
Instructions: Work any three problems.

(1) (a) Prove or disprove: If $\{f_k\}_{k=1}^{\infty}$ is uniformly integrable over a set E and $\{f_n\} \to f$ pointwise $a.e.$ on E, then f is integrable over E.

(b) Let $\mathcal{F}$ be a family of functions, each of which is integrable over E. Show that $\mathcal{F}$ is uniformly integrable over E if and only if for each $\varepsilon > 0$, there is a $\delta > 0$ such that for each $f \in \mathcal{F}$,

$$if\ A \subseteq E\ is\ measurable\ and\ m(A) < \delta, then\ \left|\int_A f\right| < \varepsilon.$$

(2) (a) Let $\{f_n\} \to f$ in measure on E and g be a measurable function on E that is finite *a.e.* on E. Show that $\{f_n\} \to g$ in measure on E if and only if $f = g$ *a.e.* on E.

(b) Let f and each f_n be integrable and $\int |f_n - f| \, dm \to 0$. Show that $f_n \to f$ in measure.

(3) A real-valued function f defined on an interval $[a, b]$ satisfies a Lipschitz-condition with constant M if $|f(y) - f(x)| \leq M|y - x|$ for all $x, y \in [a, b]$. Prove that f satisfies a Lipschitz condition with constant M if and only if

(i) f is absolutely continuous on $[a, b]$, and

(ii) $|f'(x)| \leq M \; m - a.e.$

(4) (a) Let f be integrable over $[0,1]$. Show that

$$exp\Big[\int_0^1 f(x)\ dx\Big] \leq \int_0^1 exp(f(x))\ dx$$

(b) Compute $TV_{[0,50]}(e^x)$, the total variation of e^x on the interval $[0,50]$.

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