King Fahd University of Petroleum and Minerals
Department of Mathematics & Statistics

Math 531 (Real Analysis) Major Exam I Spring 2017-2018(172)- 120 minutes

ID#:________________ NAME:______________________________

Notation: $\mathbb{R}$ = the real numbers, $\mathbb{N}$ = the natural numbers, m = Lebesgue measure.
Instructions: Give a detail solution.

---

(1) (a) A collection $\mathcal{C}$ of subsets of $\mathbf{X}$ is an algebra with the following property: if $E_n \in \mathcal{C}$ and $E_n \subseteq E_{n+1}$, $n = 1, 2, \cdots$, then $\bigcup_{n=1}^{\infty} E_n \in \mathcal{C}$. Prove that $\mathcal{C}$ is a σ-algebra.

(b) Let $f : [a, b] \to [-\infty, \infty]$ be a measurable function. Suppose that f takes the value $\pm\infty$ only on a set of (Lebesgue) measure zero. Prove that for any $\varepsilon > 0$ there is a positive number M such that $|f| < M$, except on a set of measure less than ε.

(2) (a) Assume that m^* is an outer measure on $2^{\mathbf{X}}$ and pick $A \subseteq C \subseteq \mathbf{X}$. Show that if a set B is m^*-measurable and satisfies $A \subseteq B$ and $m^*(A) = m^*(B)$, then $m^*(C) = m^*(B \cup C)$.

(b) Let $\{f_n\}$ be a sequence of Lebesgue measurable functions on a set $E \subset \mathbb{R}$, where E is of finite Lebesgue measure ($m(E) < \infty$). Suppose that there is $M > 0$ such that $|f_n(x)| \leq M$ for $n \geq 1$ and for all $x \in E$, and suppose that $\lim_{n\to\infty} f_n(x) = f(x)$ for each $x \in E$. Use Egoroff's theorem to prove that

$$\int_E f(x)\, dx = \lim_{n\to\infty} \int_E f_n(x)\, dx.$$

(3) Let $f(x)$ be a real-valued Lebesgue integrable function on $[0,1]$.

(a) Prove that if $f > 0$ on a set $E \subset [0,1]$ of positive measure, then

$$\int_E f(x)\, dx > 0.$$

(b) Prove that if

$$\int_{[0,t]} f(x)\, dx = 0, \quad \textit{for each } t \in [0,1],$$

then $f(t) = 0$ for almost all $t \in [0,1]$.

(4) (a) Suppose $f : \mathbb{R} \to \mathbb{R}$ is measurable. Use the definition of measurability to prove that for every integer $n \geq 2$, the function f^n defined by $(f^n)(x) = (f(x))^n$ is also measurable. [Note: In your proof you should not use the fact that the product of measurable functions is measurable.]

(b) Set $f_n(x) = \dfrac{n + \cos(nx)}{2n+1}$, for $x \in (0, +\infty)$ and $n \in \mathbb{N}$. Evaluate with proof $\lim\limits_{n\to\infty} \int\limits_0^n f_n(x)dx$.

(5) (a) Show that the monotone convergence theorem may not hold for decreasing sequences of functions.

(b) Let $Q = [0,1] \times [0,1] = \{(x,y) \mid 0 \leq x \leq 1, 0 \leq y \leq 1\}$ and $f : Q \to \mathbb{R}$. Assume that $x \to f(x,y)$ is a measurable function for each fixed value of $y \in [0,1]$ and the partial derivative $\dfrac{\partial f}{\partial y}$ exists. Suppose there is a function g that is integrable over $[0,1]$ such that

$$\left|\frac{\partial f}{\partial y}(x,y)\right| \leq g(y), \quad for\ all\ (x,y) \in Q.$$

Prove that

$$\frac{d}{dy}\left[\int_0^1 f(x,y)\ dx\right] = \int_0^1 \frac{\partial f}{\partial y}(x,y)\ dx, \quad for\ all\ y \in [0,1].$$

Dr. M. R. Alfuraidan