King Fahd University of Petroleum and Minerals
Department of Mathematics & Statistics
Math 470 Final Exam
The Second Semester of 2017-2018 (172)
Time Allowed: 120mn

Name: ID number:

Textbooks are not authorized in this exam

Problem #	Marks	Maximum Marks
1		10
2		15
3		15
4		15
5		15
6		15
7		15
Total		100

Problem 1: (10 pts) Consider a wave equation on the entire spatial axis

$$u_{tt} = 4u_{xx}, \quad -\infty < x < \infty, \quad t > 0, \tag{1}$$

with initial data

$$u_t(x,0) = 0, \quad u(x,0) = \begin{cases} \cos(\pi x), & \text{for } 0 < x < 3, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

1.) Give the solution of the Cauchy problem.
2.) Write the solution explicitly when $t = 1/2$.

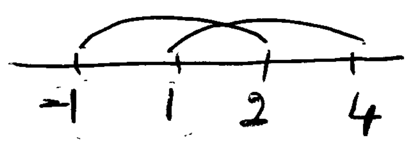
Solution

1) $u(x,t) = \frac{1}{2}\left[\varphi(x-2t) + \varphi(x+2t)\right]$ (d'Alembert formula)

2) $u(x,\frac{1}{2}) = \frac{1}{2}\left[\varphi(x-1) + \varphi(x+1)\right]$

$0 < x-1 < 3 \iff 1 < x < 4$

$0 < x+1 < 3 \iff -1 < x < 2$



Thus, $u(x,\frac{1}{2}) = \begin{cases} \cos(\pi(x+1)), & x \in (-1,1) \\ \cos(\pi(x-1)) + \cos(\pi(x+1)), & x \in (1,2) \\ \cos\pi(x-1), & x \in (2,4), \\ 0, & \text{otherwise} \end{cases}$

that is, $u(x,\frac{1}{2}) = \begin{cases} -\cos\pi x & x \in (-1,1) \\ -2\cos\pi x & x \in (1,2) \\ -\cos\pi x & x \in (2,4) \\ 0, & x \in (-\infty,-1) \cup (4,\infty) \end{cases}$

Problem 2: (15 pts)Consider the nonhomogeneous IBVP

$$\begin{aligned} &u_t = u_{xx} + F(x,t), \quad 0 < x < 1,\ t > 0 \\ &u(0,t) = u(1,t) = 0, \quad t \geq 0 \\ &u(x,0) = 1 \quad 0 \leq x \leq 1. \end{aligned}$$

Assume u and F are in the forms

$$u(x,t) = \sum_{n=1}^{\infty} A_n(t)\sin(n\pi x) \quad \text{and} \quad F(x,t) = \sum_{n=1}^{\infty} B_n(t)\sin(n\pi x).$$

1.) Express $A_n(t)$ in terms of u; and $B_n(t)$ in terms of F.
2.) Write the ODE satisfied by $A_n(t)$ and $B_n(t)$.
3.) Compute $A_n(0)$ and find the explicit expression of $A_n(t)$, for $t \geq 0$.

Solution

1) $A_n(t) = 2\int_0^1 u(\xi,t)\sin(n\pi\xi)\,d\xi$; $B_n(t) = 2\int_0^1 F(\xi,t)\sin(n\pi\xi)\,d\xi$

2) $A_n'(t) = 2\int_0^1 u_t(\xi,t)\sin(n\pi\xi)\,d\xi = 2\int_0^1 [u_{xx}(\xi,t) + F(\xi,t)]\sin n\pi\xi\,d\xi$

But, $\int_0^1 u_{xx}(\xi,t)\sin(n\pi\xi)\,d\xi = -(n\pi)^2\int_0^1 u(\xi,t)\sin(n\pi\xi)\,d\xi$
(by integration by parts, twice)

$\Rightarrow A_n'(t) + 2(n\pi)^2 A_n(t) = B_n(t)$

3) $A_n(0) = 2\int_0^1 u(\xi,0)\sin(n\pi\xi)\,d\xi = 2\int_0^1 \sin(n\pi\xi)\,d\xi = 2\left(\frac{1-(-1)^n}{n\pi}\right)$

But, $A_n(t) = e^{-2(n\pi)^2 t} A_n(0) + e^{-2(n\pi)^2 t}\int_0^t B_n(s)\,e^{2(n\pi)^2 s}\,ds$

$$A_n(t) = e^{-2(n\pi)^2 t}\cdot\frac{2(1-(-1)^n)}{n\pi} + \int_0^t B_n(s)\,e^{-2(n\pi)^2(t-s)}\,ds$$

Problem 3: (15 pts) Let Ω be a bounded domain of $\mathbb{R}^3$, $\partial\Omega$ a piecewise smooth closed boundary surface. Let x be an interior point of Ω and let $B(x,\varepsilon) = \{y \in \mathbb{R}^3,\ |y-x| < \varepsilon\}$ be the open ball of center x and radius $\varepsilon > 0$. Let $S(x,\varepsilon) = \partial B = \{y \in \mathbb{R}^3,\ |y-x| = \varepsilon\}$ be the sphere of center X and radius ε. Let $u \in \mathcal{C}^2(\overline{\Omega})$. Show that

$$\lim_{\varepsilon\to 0} \iint_{S(x,\varepsilon)} \left[u(y)\frac{\partial}{\partial n}\left(\frac{1}{|y-x|}\right) - \frac{1}{|y-x|}\frac{\partial u(y)}{\partial n}\right] d\sigma_y = 4\pi u(x). \tag{3}$$

where $d\sigma_y$ is the differential element of surface area on $S(x,\varepsilon)$; $n(y)$ is the outer normal vector to $S(x,\varepsilon)$ and $n(y) = -\frac{y-x}{|y-x|}$ for any point $y \in S(x,\varepsilon)$.

$\left(\underline{\textbf{Hint:}}\text{ we have } \frac{\partial}{\partial n}\left(\frac{1}{|y-x|}\right) = \frac{1}{\varepsilon^2}, \text{ for any } y \in S(x,\varepsilon)\right)$.

<u>Solution</u>

$y \in S \Rightarrow \frac{1}{|y-x|} = \frac{1}{\varepsilon}$ and $\frac{\partial}{\partial n}\left(\frac{1}{|y-x|}\right) = \frac{1}{\varepsilon^2}$

Thus, $\iint_S \left[u(y)\frac{\partial}{\partial n}\left(\frac{1}{|y-x|}\right) - \frac{1}{|y-x|}\frac{\partial u(y)}{\partial n}\right] d\sigma_y = \iint_S \left[\frac{1}{\varepsilon^2}u(y)\,d\sigma_y - \frac{1}{\varepsilon}\frac{\partial u(y)}{\partial n}\right] d\sigma_y$

$$= \underbrace{\frac{1}{\varepsilon^2}\iint_S u(x)\,d\sigma_y}_{\frac{1}{\varepsilon^2}(4\pi\varepsilon^2)\,u(x)} + \frac{1}{\varepsilon^2}\iint_S [u(y)-u(x)]\,d\sigma - \frac{1}{\varepsilon}\iint_S \frac{\partial u(y)}{\partial n}$$

But, $\left|\frac{1}{\varepsilon^2}\iint_S [u(y)-u(x)]\,d\sigma_y - \frac{1}{\varepsilon}\iint_S \frac{\partial}{\partial n}u(y)\,d\sigma_y\right| \le \frac{1}{\varepsilon^2}(4\pi\varepsilon^2)\max_{y\in S}|u(y)-u(x)|$
$+ \frac{1}{\varepsilon}(4\pi\varepsilon^2)\max_{y\in S}\left|\frac{\partial u(y)}{\partial n}\right|$

But, $u(y) \to u(x)$ when $\varepsilon \to 0$

and $\left|\frac{\partial}{\partial n}u(y)\right|$ is bounded for $y \in S$

$\Rightarrow \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2}\iint_S (u(y)-u(x))\,d\sigma_y - \frac{1}{\varepsilon}\iint_S \frac{\partial}{\partial n}u(y)\,d\sigma_y = 0$

and $\lim_{\varepsilon\to 0} \iint_S \left[u(y)\frac{\partial}{\partial n}\left(\frac{1}{|y-x|}\right) - \frac{1}{|y-x|}\frac{\partial u(y)}{\partial n}\right] d\sigma_y = 4\pi u(x)$

Problem 4: (15 pts) Consider the boundary value problem

$$\begin{aligned} \nabla^2 u &= 0 \quad \text{in } D \\ u &= f \quad \text{on } \partial D \end{aligned}$$

where D is a simply-connected 2D region with piecewise smooth boundary ∂D.
1.) State the Maximum Principle for u on D. If $f = 3$ at each point on the boundary ∂D, what is u on D?
2.) State the Mean Value Property of u on the circle $C = \{(x, y) : x^2 + y^2 = R\}$. Use this result to find $u(0,0)$ if on the boundary u takes the values

$$u(R, \theta) = \begin{cases} 1, & \text{if } -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}, \\ 0, & \text{if } \frac{\pi}{2} \le \theta \le \pi, \\ -1, & \text{if } \pi \le \theta \le \frac{3\pi}{2}. \end{cases}$$

Solution

1) If u is non constant and continuous on $\bar{D}$, then it assumes its maximum and minimum values on D only at points of ∂D

Let $v = u - 3$. Then, $\nabla^2 v = 0$ and $v = 0$ on ∂D.

The Maximum principle on $v \Rightarrow \max_{\bar{D}} v = \min_{\bar{D}} v = 0$

So, $v(x \;\; 0$ on $\bar{D}$, $\underline{u = 3 \text{ on } D}$

2) $u(0,0) = \frac{1}{2\pi R} \int_{C} u(\xi, \eta)\, ds$ (Mean value property of u on C)

$s = R\theta \Rightarrow ds = R\,d\theta \Rightarrow u(0,0) = \frac{1}{4\pi R} \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} u(R,\theta)\, R\, d\theta$

$$\Rightarrow u(0,0) = \frac{1}{2\pi R}\Big[\underbrace{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} R\,d\theta}_{R\pi} + \underbrace{\int_{\frac{\pi}{2}}^{\pi} 0 \times R\,d\theta}_{= 0} + \underbrace{\int_{\pi}^{\frac{3\pi}{2}} (-R)\,d\theta}_{-\frac{\pi}{2}R}\Big]$$

$$= \frac{1}{2\pi R}\left(R\frac{\pi}{2}\right) = \frac{1}{4}$$

Problem 5: (15 pts) Solve the Laplace equation on the quarter unit disc

$$\nabla^2 v(r,\theta) = v_{rr} + \frac{1}{r}v_r + \frac{1}{r^2}v_{\theta\theta} = 0,$$

with boundary conditions

$$v(1,\theta) = g(\theta), \quad v(0,\theta) \text{ bounded}, \quad -\frac{\pi}{2} < \theta < \ldots,$$
$$v(r,-\frac{\pi}{2}) = 0, \quad v(r,\pi) = 0, \quad 0 < r < 1.$$

Solution

$$v(r,\theta) = R(r)T(\theta), \quad -\frac{T''}{T} = \frac{r^2(R'' + \frac{1}{r}R')}{R} = -\lambda$$

$$\begin{cases} T'' - \lambda T = 0 \\ T(-\frac{\pi}{2}) = T(0) = 0 \end{cases} \quad \text{and} \quad r^2R'' + rR' + \lambda R = 0$$

- $\lambda = \alpha^2$, $T(\theta) = c_1 e^{\alpha\theta} + c_2 e^{-\alpha\theta}$

$$\begin{matrix} T(-\pi/2) = 0 \Rightarrow \\ T(0) = 0 \Rightarrow \end{matrix} \begin{cases} c_1 e^{-\frac{\alpha\pi}{2}} + c_2 e^{\frac{\pi}{2}\alpha} = 0 \\ c_1 + c_2 = 0 \end{cases} \Rightarrow c_1 = c_2 = 0$$

- $\lambda = 0$, $T(\theta) = c_1\theta + c_2$

$$\begin{matrix} T(-\pi/2) = 0 \Rightarrow \\ T(0) = 0 \Rightarrow \end{matrix} \begin{cases} -\frac{c_1\pi}{2} + c_2 = 0 \\ c_2 = 0 \end{cases} \Rightarrow c_1 = c_2 = 0$$

- $\lambda = -\alpha^2$, $T(\theta) = c_1 \cos\alpha\theta + c_2 \sin\alpha\theta$

$T(0) = 0 \Rightarrow c_1 = 0$

$T(-\frac{\pi}{2}) = 0 \Rightarrow \sin(\frac{\pi}{2}\alpha) = 0, \quad \alpha = 2n$

$$r^2R'' + rR' - (2n)^2R = 0, \quad R_n(r) = c_1 r^{2n} + c_2 r^{-2n}$$

$v(0,\theta)$ bounded $\Rightarrow c_2 = 0$, $R_n = c\, r^{2n}$

Thus, $$v(r,\theta) = \sum_{n=1}^{\infty} B_n \sin(2n\theta)\, r^{2n}$$

$$v(1,\theta) = g(\theta) \Rightarrow g(\theta) = \sum_{n=1}^{\infty} B_n \sin 2n\theta, \quad B_n = \frac{4}{\pi}\int_{-\frac{\pi}{2}}^{0} \sin(2n\xi)g(\xi)\,d\xi$$

Problem 6: (15 pts) We know that the Laplace equation for the upper half-plane with boundary conditions on the x-axis

$$u_{xx} + u_{yy} = 0, \ u(x,0) = f(x), \ -\infty < x < \infty, \ y > 0, \tag{4}$$

has the solution

$$u(x,y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y f(\xi)}{y^2 + (\xi - x)^2} d\xi. \tag{5}$$

Consider the equation in the upper left quadrant with homogeneous boundary conditions on the positive y-axis:

$$v_{xx} + v_{yy} = 0, \ v(x,0) = g(x), \ v(0,y) = 0, \ x < 0, \ y > 0. \tag{6}$$

Use the solution of (4) to find an integral expression of the solution of (6), given on the negative x-axis.

Solution

Let $\begin{cases} v_{xx} + v_{yy} = 0 \\ v(x,0) = \tilde{g}(x) \\ v(0,y) = 0 \end{cases}$, $\tilde{g}(x) = \begin{cases} g(x), & x < 0 \\ -g(-x), & x > 0 \end{cases}$

This problem is defined on the entire real line $x \in (-\infty, \infty)$.

Its solution can be written as (5), that is,

$$v(x,y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{\tilde{g}(\xi)}{y^2 + (\xi - x)^2} d\xi$$

$$= \frac{y}{\pi} \left[\int_{-\infty}^{0} \frac{g(\xi)}{y^2 + (\xi - x)} d\xi + \underbrace{\int_{0}^{\infty} \frac{-g(-\xi)}{y^2 + (\xi - x)^2} d\xi}_{\text{use } = \int_{-\infty}^{0} \frac{-g(\xi)}{y^2 + (\xi + x)^2} d\xi} \right]$$

$$\Rightarrow v(x,y) = \frac{y}{\pi} \int_{-\infty}^{0} \left(\frac{1}{y^2 + (\xi - x)^2} - \frac{1}{y^2 + (\xi - x)^2} \right) g(\xi) \, d\xi$$

Problem 7: (15 pts) Let Ω be an open bounded domain with a piecewise smooth boundary $\partial\Omega$. Consider the Robin problem for the Laplace equation

$$u - \nabla^2 u = 0 \qquad \text{in } \Omega,$$
$$\frac{\partial u}{\partial n} + \alpha u = \beta \quad \text{on } \partial\Omega.$$

where $\alpha > 0$ and $\beta \in \mathbb{R}$. Prove that the solution u is unique.

Solution

Let u and v be two solutions of the problem.

Let $w = u - v$.

We have
$$\begin{cases} w - \nabla^2 w = 0 & \text{in } \Omega & (1) \\ \frac{\partial}{\partial n} w + \alpha w = 0 & \text{on } \partial\Omega, & (2) \end{cases}$$

Multiply (1) by w and integrate over $\bar{\Omega}$.

$$\int_{\Omega} |w|^2 dx + \int_{\bar{\Omega}} w \nabla^2 w \, dx = 0 \qquad (3)$$

The Green's formula is $\int_{\partial\Omega} w \frac{\partial w}{\partial n} ds = \int_{\bar{\Omega}} \left(w \nabla^2 w + |\nabla w|^2 \right) dx$

(3)
$$\int_{\Omega} |w|^2 dx + \int_{\bar{\Omega}} |\nabla w|^2 dx = \int_{\partial\Omega} w \frac{\partial w}{\partial n} ds \qquad (4)$$

$$(2) \Rightarrow \quad \frac{\partial w}{\partial n} = -\alpha w$$

$$(4) \Rightarrow \quad \int_{\bar{\Omega}} |\nabla w|^2 dx = -\int_{\partial\Omega} \alpha |w|^2 ds \leq 0$$

$$\int_{\bar{\Omega}} |w|^2 dx + \int_{\bar{\Omega}} |\nabla w|^2 dx = 0 \Rightarrow w = 0 \text{ in } \bar{\Omega}$$

that is, $u = v$ in $\bar{\Omega}$.