MATH411, Section 1
Spring 2018, Term 172

Exam 1

ID:

Name:

Answer the following questions.

1. (10 pts) For $x, y \in \mathbb{R}^n$ show that

$$\|x+y\|\|x-y\| \leq \|x\|^2 + \|y\|^2$$

and the equality holds if and only if $\langle x, y\rangle = 0$.

2. (10 pts) Prove that a closed subspace of a complete metric space is complete.

3. (10 pts) Let X be a metric space. Show that $(\partial S)^\circ = \emptyset$ where S is an open subset of X.

4. (10 pts) Let $f : X \to Y$ be a function from a metric space X to another metric space Y. Show that if f is one-to-one, then

$$A = f^{-1}(f(A)), \quad \text{for any } A \subseteq X.$$

5. (10 pts) Using "$\epsilon - \delta$" definition of the limit, show that

$$\lim_{(x,y)\to(5,-1)} (x - 5y) = 10$$

6. (10 pts) Show that the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined as

$$f(x,y) = \begin{cases} y^2 + x^3 \sin\left(\frac{1}{x}\right) & \text{if} \quad x \neq 0, \\ y^2 & \text{if} \quad x = 0 \end{cases}$$

is continuous at $(0,0)$.

7. (10 pts) A subset S of a metric space X is said to be *connected* if it cannot be written as $S = (S \cap A) \cup (S \cap B)$ where A and B are disjoint nonempty open subsets of X. Suppose now that X and Y are two metrics spaces and $F : X \to Y$ is a continuous function. Show that if $S \subset X$ is connected then so is $F(S)$.