King Fahd University of Petroleum & Minerals

Department of Mathematics and Statistics

Math 345: Modern Algebra I

Final Exam, Spring Semester 172 (150 minutes)

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Show FULL details.

Q1. (10 points)

(a) Show that if R is a *unital* commutative ring, then every maximal ideal of R is prime.

(b) Give an example of a maximal ideal of a *non-unital* commutative ring which is not prime.

Q2. (10 points) Show that:

(a) No integer of the form

$$111,111,111,\cdots,111$$

is prime.

(b) $q(x) = 30x^n - 91$ $(n > 1)$ has not *rational* roots.

Q3. (10 points) Let D be an integral domain. Show that $D[x]$ is a PID if and only if D is a field.

Q4. (10 points) Let p be a prime positive integer.

(a) $\Phi_p(x) = x^{p-1} + x^{p-2} + \cdots + x + 1$ is irreducible over $\mathbb{Q}$.

(b) There exists a field of order p^2.

Q5. (10 points) Let R be a PID. Show that:

(a) Every irreducible element is prime.

(b) R is a UFD.

Q6. (10 points) Show that:

(a) A_4 has no subgroup of order 6.

(b) A_8 has a *cyclic* subgroup of order 4 as well as a *non-cyclic* subgroup of order 4.

Q7. (10 points)

(a) Give an example of an *infinite* field of characteristic 3.

(b) Show that if F is a finite field with $|F| - 1$ a prime number $p \geq 3$, then F has exactly two subfields.

Q8. (30 points) Prove or disprove:

(a) If I is a maximal ideal of R, then $I[x]$ is a maximal ideal of $R[x]$.

(b) For any commutative ring R, we have $U(R[x]) = U(R)$.

(c) For every even integer n, the dihedral group D_n contains a subgroup of order 4.

BONUS: (5 points) Show that there are (up to isomorphism) exactly 2 groups of order 99.

GOOD LUCK