King Fahd University of Petroleum & Minerals

Department of Mathematics and Statistics

Math 345: Group Theory

Second Exam, Spring Semester 172 (120 minutes)

Jawad Abuhlail

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Show FULL details.

Q1. (10 points) Let G be a group and let $Z(G)$ be the center of G. Show that:

(a) If $G/Z(G)$ is cyclic, then G is Abelian.

(b) $G/Z(G) \simeq Inn(G)$.

Q2. **(10 points)** Let G be a group and $N \leq G$ a subgroup.

(a) Show that N is a normal subgroup of G if and only if there exists a group G' and a group homomorphism $f : G \to G'$ such that $N = \text{Ker}(f)$.

Q3. (15 points) Give examples of 5 groups of order 12, no two of which are isomorphic. Show why no two are isomorphic.

Q4. (10 points) Let G be a group and N a normal subgroup of G. Show that:

(a) for any subgroup K of G, we have

$$K/(K \cap N) \simeq KN/N.$$

(b) every subgroup of G/N has the form H/N for some subgroup $H \leq G$.

Q5. (15 points) Show that there are:

(a) two Abelian groups of order 108 that contain exactly 1 subgroup of order 3;

(b) two Abelian groups of order 108 that contain exactly 4 subgroups of order 3;

(c) two Abelian groups of order 108 that contain exactly 13 subgroups of order 3.

Q6. (10 points) Show that:

(a) every finite integral domain is a field.

(b) any finite field has order p^n, where p is a prime and $n \in \mathbb{Z}^+$.

Q7. (30 points) Prove or disprove:

(a) $\mathbb{Z} \oplus \mathbb{Z}$ is a cyclic group.

(b) Let G be a group, $H, K \leq G$ be subgroups of G such that $G = HK$ and $H \cap K = \{e\}$. Then $G \simeq H \times K$.

(c) If G is a group of a prime-power order and $a \in G$ is of maximum order, then $G = <a> \times K$ for some subgroup K of G.

GOOD LUCK