King Fahd University of Petroleum & Minerals
Department of Mathematics & Statistics
Math 321 Final Exam (172)
Time Allowed: 180 Minutes ——(V1)

Name: KEY ID#:

Instructor: Sec #: Serial #:

- Mobiles are not allowed in this exam.
- Answers should be neat, clear, and legible with all clear steps.
- Write your final answers in four decimal places

Written	Marks	Max Marks
1		15
2		13
3		15
4		15
5		10
6		10
Total		78

V1

1	B
2	E
3	C
4	D
5	E
6	B
7	A

MCQ	Answer	Marks
1		
2		
3		
4		
5		
6		
7		
Total		42
	Grand Total out of 120	

V2

1	E
2	D
3	B
4	C
5	E
6	D
7	A

Q1 (a) Find the LU factorization of the matrix $A = \begin{bmatrix} 2 & 2 & 6 & 8 \\ 2 & 3 & 2 & 5 \\ 1 & 1 & -1 & 2 \\ -1 & -1 & 1 & 1 \end{bmatrix}$. (10 pts)

(b) Solve the system $LUx = b$, where $x = (x_1, x_2, x_3, x_4)$ and $b = (0, 3, 4, 2)$. (5 pts)

$E_2 - E_1 \to E_2$, $E_3 - \frac{1}{2}E_1 \to E_3$, $E_4 + \frac{1}{2}E_1 \to E_4$: $\begin{bmatrix} 2 & 2 & 6 & 8 \\ 0 & 1 & -4 & -3 \\ 0 & 0 & -4 & -2 \\ 0 & 0 & 4 & 5 \end{bmatrix}$ $\quad E_4 + E_3 \to E_4$ $\begin{bmatrix} 2 & 2 & 6 & 8 \\ 0 & 1 & -4 & -3 \\ 0 & 0 & -4 & -2 \\ 0 & 0 & 0 & 3 \end{bmatrix} = U$

$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & 1 & 0 \\ -\frac{1}{2} & 0 & -1 & 1 \end{bmatrix}$. Let $Ux = y$, then $Ly = b$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & 1 & 0 \\ -\frac{1}{2} & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 4 \\ 2 \end{bmatrix} \Rightarrow y_1 = 0, \quad y_2 = 3, \quad y_3 = 4, \quad y_4 = 2+4 = 6$$

$$\begin{bmatrix} 2 & 2 & 6 & 8 \\ 0 & 1 & -4 & -3 \\ 0 & 0 & -4 & -2 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 4 \\ 6 \end{bmatrix}$$

$\underline{x_4 = 2}$,

$-4x_3 = 4 + 2x_4 = 8$

$\underline{x_3 = -2}$

$x_2 = 3 + 4x_3 + 3x_4 = 3 - 8 + 6 = \underline{1}$

$2x_1 = 0 - 2x_2 - 6x_3 - 8x_4 = -2 + 12 - 16$

$\underline{x_1 = -3}$

$(-3, 1, -2, 2)$

Q2 Find the first two iterations of the Gauss-Seidel method for the following linear system using $x^{(0)} = (0,0,0)$

$$\begin{aligned} 3x_1 - x_2 + x_3 &= 1 \\ 3x_1 + 6x_2 + 2x_3 &= 0 \\ 3x_1 + 3x_2 + 7x_3 &= 4 \end{aligned}$$

(13 pts)

$$x_1 = [1 + x_2 - x_3]/3$$

$$x_2 = [-3x_1 - 2x_3]/6$$

$$x_3 = [4 - 3x_1 - 3x_2]/7$$

$$x_1^{(1)} = \frac{1}{3}, \quad x_2^{(1)} = -1/6, \quad x_3^{(1)} = \frac{4 - 1 + \frac{1}{2}}{7} = \frac{1}{2}$$

$$= 0.3333 \qquad = 0.1667 \qquad = 0.5000$$

$$x_1^{(2)} = [1 - \frac{1}{6} - \frac{1}{2}]/3 = \frac{6-1-3}{18} = \frac{2}{18} = \frac{1}{9}$$

$$= 0.1111$$

$$x_2^{(2)} = [-\frac{1}{3} - 1]/6 = \frac{-1-3}{18} = \frac{-4}{18} = -0.2222$$

$$x_3^{(2)} = [4 - \frac{1}{3} + \frac{2}{3}]/7 = \frac{13}{21}$$

$$= 0.6190$$

Q3 (a) Use centered-difference formulas with $h = 0.2$ for the first and second derivatives to write the finite difference equations for the BVP: (12 pts)

$$\cancel{y'' + 3y' - 2y = 2x + 3},\quad 0 \le x \le 1,\quad y(0) = 2,\quad y(1) = 1$$

(b) Rewrite the system of equations in Part (a) in matrix form. (3 pts)

$$-y'' - 3x^2 y' + 2xy = -(2x+3)$$

$$-\frac{w_{i+1} + 2w_i - w_{i-1}}{h^2} - 3x_i^2 \frac{w_{i+1} - w_{i-1}}{2h} + 2x_i w_i = -(2x_i + 3)$$

$$-w_{i+1} + 2w_i - w_{i-1} - \frac{3h}{2} x_i^2 \left[w_{i+1} - w_{i-1}\right] + 2x_i h^2 w_i = -(2x_i+3)h^2$$

$$\left[-1 + \frac{3h}{2} x_i^2\right] w_{i-1} + \left[2 + 2x_i h^2\right] w_i + \left[-1 - \frac{3h}{2} x_i^2\right] w_{i+1} = -(2x_i+3)h^2$$

$i = 1, 2, 3, 4$

$$\begin{bmatrix} 2.016 & -1.012 & 0 & 0 \\ -0.952 & 2.032 & -1.048 & 0 \\ 0 & -0.892 & 2.048 & -1.108 \\ 0 & 0 & -0.808 & 2.064 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} = \begin{bmatrix} 1.840 \\ -0.1520 \\ -0.1680 \\ 1.008 \end{bmatrix}$$

Q4 Consider the following set of data $\{(1,1),(2,1.5),(3,1.75),(4,2),(5.25)\}$. Use the least square approach to write the system of linear equations to compute the constants a_o, a_1 and a_2 such that $y = a_0 + a_1x + a_2x^2$ is the least square polynomial of degree 2 for the data. (Do not solve the system). (15 pts)

x	y	xy	x^2	x^2y	x^3	x^4
1	1	1	1	1	1	1
2	1.5	3	4	6	8	16
3	1.75	5.25	9	15.75	27	81
4	2	8	16	32	64	256
5	0.25	1.25	25	6.25	125	625
15	6.5	18.5	55	61	225	979

$$5a_0 + 15a_1 + 55a_2 = 6.5$$
$$15a_0 + 55a_1 + 225a_2 = 18.5$$
$$55a_0 + 225a_1 + 979a_2 = 61$$

Q5 Write a Matlab code that does the following:

(a) Determine the linear least square polynomial for the a set of data of the form (6 pts)

$$\{(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), (x_5, y_5), (x_6, y_6)\}.$$

(b) Plot the set of data and its linear fit in the same figure window. (4 pts)

```
clear all
sumx=0; sumx2=0;
sumy=0; sumxy=0;
x=[x1 x2 x3 x4 x5 x6];
y=[y1 y2 y3 y4 y5 y6];
n = length(x);
for i=1:n
    sumx= sumx + x(i);
    sumx2 = sumx2 + x(i)^2;
    sumy = sumy + y(i);
    sumxy = sumxy + x(i)*y(i);
end
A=[n sumx; sumx sumx2];
b=[sumy; sumxy];
a = inv(A)*b;
a0 = a(1); a1 = a(2);
P=a0 + a1*x
plot(x,y,x,P)
```

Q6 Write a Matlab code to approximate the solution of the initial value problem

$$y' = t^{-2}(\sin(2t) - 2ty), \quad 1 \le t \le 2,$$

$$y(1) = 2$$

using Euler method with $h = 0.2$.

Also include the commands to plot the numerical solution and the exact solution

$$y = \frac{4 + \cos 2 + \cos 2t}{2t^2}$$

in the same window (10 pts)

```
clear all
f=@(t,y) t^-2 *(sin(2*t)-2*t*y);
g=@ t (4 + Cos(2) + Cos(2*t))./(2* t.^2);
a=1; b=2; h=0.2; alf=2; n= (b-a)/h;
t(1)=a; w(1)=alf;

for i=1:n
    w(i+1) = w(i)+ h * f(t(i), w(i));
    t(i+1) = a + i * h;
end

plot(t, w, t, g(t))
```