King Fahd University of Petroleum & Minerals
Department of Mathematics & Statistics
Math 321 Major Exam 2 (172)

Time Allowed: 120 Minutes

Name: KEY ID#: ______

Instructor: ______ Sec #: ______ Serial #: ______

- Mobiles are not allowed in this exam.
- Answers should be neat, clear, and legible.
- Show all steps
- Write your answers in four significant digits

Question #	Marks	Maximum Marks
1		12
2		15
3		12
4		12
5		13
6		8
7		8
Total		80

Q1 The Natural cubic splines for the data $x = [2, 3, 4]$ and $f(x) = [3, 4, 6]]$ are

$$S_0(x) = a_0 + \tfrac{3}{4}(x-2) + c_0(x-2)^2 + d_0(x-2)^3$$

$$S_1(x) = a_1 + \tfrac{3}{2}(x-3) + c_1(x-3)^2 + d_1(x-3)^3$$

Use necessary conditions for Natural cubic splines to find unknown constants. (12 pts)

$S_0(2) = f(2) \Rightarrow \underline{a_0 = 3}$

$S_1(3) = f(3) \Rightarrow \underline{a_1 = 4}$

$S_0(3) = f(3) \Rightarrow a_0 + \frac{3}{4} + c_0 + d_0 = 4 \rightarrow (*)$

$S_0'(x) = \frac{3}{4} + 2c_0(x-2) + 3d_0(x-2)^2$

$S_0''(x) = 2c_0 + 6d_0(x-2)$

$S_1''(x) = 2c_1 + 6d_1(x-3)$

$S_0''(2) = 0 \Rightarrow \underline{c_0 = 0}$

$S_0''(3) = S_1''(3) \Rightarrow 2c_0 + 6d_0 = 2c_1 \rightarrow (**)$

From $(*)$ $d_0 = 4 - 3 - \frac{3}{4} = 1 - \frac{3}{4} = \frac{1}{4}$

From $(**)$ $c_1 = 3d_0 = \frac{3}{4}$

$\underline{d_0 = \frac{1}{4}}$ $\underline{c_1 = \frac{3}{4}}$

$S_1''(4) = 0 \Rightarrow 2c_1 + 6d_1 = 0$

$d_1 = -\frac{2}{6}c_1 = -\frac{1}{3} \cdot \frac{3}{4} = -\frac{1}{4}$

$\underline{d_1 = -\frac{1}{4}}$

Q2 Let $f(x) = \ln(x)$.

(a) Approximate $f'(2)$ using central difference formula with $h = 0.05$ and compute the actual error. (5 pts)

(b) Compute the error bound for $f'(2)$ using the formula $\frac{h^2}{6} f^{(3)}(\xi)$ for ξ in the interval $[2-h, 2+h]$. (2 pts)

(c) Approximate $f''(2)$ using central difference formula with $h = 0.05$ and compute the actual error. (5 pts)

(d) Compute the error bound for $f''(2)$ using the formula $\frac{h^2}{12} f^{(4)}(\xi)$ for ξ in the interval $[2-h, 2+h]$. (2 pts)

(a) $$f'(2) \approx \frac{f(2+h) - f(2-h)}{2h} = \frac{Ln(2.05) - Ln(1.95)}{2(0.05)}$$

$$= 0.5001$$

$$f'(x) = \frac{1}{x}, \quad f'(2) = 0.5$$

Actual Error $= |0.5001 - 0.5| = 0.00010$

(b) $$\max|f'''(x)| = \max\left|\frac{2}{x^3}\right| = 0.2697$$

Error Bound $\le \frac{(0.05)^2}{6}(0.2697) = 0.00011$

(c) $$f''(2) \approx \frac{f(2+h) - 2f(2) + f(2-h)}{h^2} = -0.250078$$
$$= -0.2501$$

$$f''(x) = \frac{-1}{x^2}, \quad f''(2) = -0.25$$

Actual Error $= |-0.250078 + 0.25| = 0.000078$
$= 0.0001$

(d) Error Bound $\le \frac{(0.05)^2}{12}\left(\frac{6}{(1.95)^4}\right) = 0.000086$
$= 0.0001$

Q3 (a) Approximate the integral $\int_0^4 \frac{2}{x^2+4}\,dx$ using composite Simpson's rule with N=6. (8 pts)

(b) Compute the actual error. (Hint: $\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}(\frac{x}{a})$) (4 pts)

x_0	x_1	x_2	x_3	x_4	x_5	x_6
0	$\frac{2}{3}$	$\frac{4}{3}$	2	$\frac{8}{3}$	$\frac{10}{3}$	4

$$h = \frac{4-0}{6} = \frac{2}{3}$$

$$\int_0^4 \frac{2}{x^2+4}\,dx \approx \frac{h}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + f(x_6)\right]$$

$$= \frac{2}{9}\left[\frac{2}{4} + \frac{8}{\frac{4}{9}+4} + \frac{4}{\frac{16}{9}+4} + \frac{8}{4+4} + \frac{4}{\frac{64}{9}+4} + \frac{8}{\frac{100}{9}+4} + \frac{2}{16+4}\right]$$

$$= 1.10705$$

$$\text{Exact} = 2 \cdot \frac{1}{2} \tan^{-1} \frac{x}{2} \Big|_0^4$$

$$= \tan^{-1}(2) = 1.10715$$

$$\text{Error} = 0.0001$$

Q4 (a) Use Euler's method with $h = 0.25$ to approximate the solution of the initial value problem

$$y' = \frac{1+t}{1+y}, \quad 1 \le t \le 2,$$
$$y(1) = 2. \qquad \text{(8 pts)}$$

(b) Compute the actual error at $t = 2$ using the exact solution

$$y(t) = \sqrt{t^2 + 2t + 6} - 1. \qquad \text{(4 pts)}$$

(a)

1 1.25 1.5 1.75 2

$w_0 = 2$

$w_1 = 2 + 0.25\left[\frac{1+1}{1+2}\right] = 2.1667$

$w_2 = 2.1667 + 0.25\left[\frac{1+1.25}{1+2.1667}\right] = 2.3443$

$w_3 = 2.3443 + 0.25\left[\frac{1+1.5}{1+2.3443}\right] = 2.5312$

$w_4 = 2.5312 + 0.25\left[\frac{1+1.75}{1+2.5312}\right] = 2.7259$

(b) Actual Error at $t=2$

$= \left| 2.7259 - \sqrt{14} + 1 \right|$

$= 0.0156$

Q5 (a) Use the RK2 method

$$w_{i+1} = w_i + hf(t_i + \tfrac{h}{2}, w_i + \tfrac{h}{2}f(t_i, w_i))$$

with $h = 0.2$ to approximate the solution of the initial value problem

$$y' = 1 + \frac{y}{t} + \left(\frac{y}{t}\right)^2, \quad 1 \le t \le 3$$

$$y(1) = 0$$

at $t_1 = 1.2$, and $t_2 = 1.4$. (10 pts)

(b) Compute the actual error at $t_2 = 1.4$ using the exact solution $y(t) = t\tan(\ln t)$. (4 pts)

(a) $t_0 = 1$, $t_1 = 1.2$, $f(t, y) = 1 + \frac{y}{t} + \left(\frac{y}{t}\right)^2$

$w_0 = 0$

$i = 1$ $\quad K = hf(t_0, w_0) = 0.2\left[1 + \frac{0}{1} + \left(\frac{0}{1}\right)^2\right] = 0.2$

$w_1 = w_0 + hf(t_0 + \frac{h}{2}, w_0 + \frac{K}{2})$

$= 0 + 0.2\left[1 + \frac{0.1}{1.1} + \left(\frac{0.1}{1.1}\right)^2\right]$

$= \boxed{0.2198}$

$i = 2$ $\quad K = hf(t_1, w_1) = 0.2\left[1 + \frac{0.2198}{1.2} + \left(\frac{0.2198}{1.2}\right)^2\right]$

$= 0.2433$

$w_2 = w_0 + hf(t_1 + \frac{h}{2}, w_1 + \frac{K}{2})$

$= 0.2198 + 0.2\left[1 + \frac{0.2198 + \frac{0.2433}{2}}{1.2 + 0.1} + \left[\frac{0.2198 + \frac{0.2433}{2}}{1.2 + 0.1}\right]^2\right] = \boxed{0.4861}$

(b) Error $= |0.4861 - 1.4\tan(\ln(1.4))| = \boxed{0.0035}$

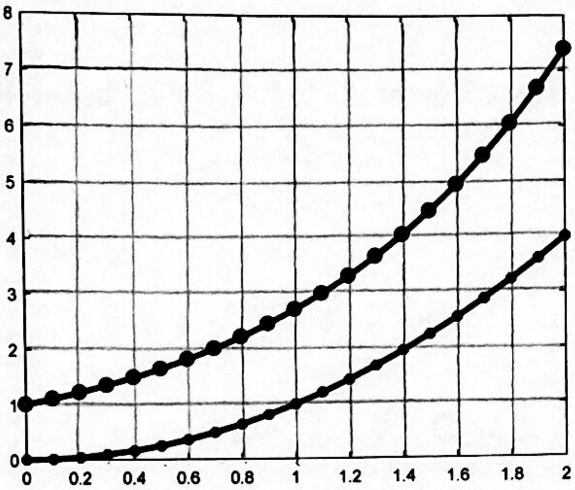

Figure 1: Graphs of $f(x) = e^x$ and g(x)=x^2

Q6 Write a Matlab code to plot the graphs same like given in the above figure. (8 pts)

```
clear all
n=20;
x = 0:0.1:2;      or      x= linspace(0,2,21)
f = @(x) exp(x);
g = @(x) x.^2;
plot(x, f(x), 'o-', x, g(x), '.-')
grid on
```

OR

```
plot(x, f(x), 'o-')
hold on
grid on
plot(x, g(x), '.-')
```

Q7 Write a Matlab code to approximate $\int_0^2 x^2\, e^{2x}\, dx$ using Trapezoidal rule. (8 pts)

```
clear all
a=0;   b=2;   h=2;
f=@(x) x.^2 * exp(2*x);

Int = h/2 *( f(a) + f(b));

Int
```