1. (a) Find the Fourier transform of $f(x) = e^{-a|x|}$, $a > 0$. 10

(b) Solve the problem using the Fourier transform

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t}, \quad -\infty < x < \infty, \quad t > 0$$ 14

$$u(x,0) = e^{-a|x|}, \quad a > 0, \quad k > 0.$$

(a) $$F(\alpha) = \int_{-\infty}^{\infty} f(x) e^{i\alpha x} dx = \int_{-\infty}^{0} e^{ax} e^{i\alpha x} dx + \int_{0}^{\infty} e^{-ax} e^{i\alpha x} dx \quad (2)$$

$$= \left. \frac{e^{(a+i\alpha)x}}{a+i\alpha} \right|_{x\to-\infty}^{0} + \left. \frac{e^{(-a+i\alpha)x}}{-a+i\alpha} \right|_{0}^{\infty} \quad (4)$$

$$= \frac{1}{a+i\alpha} + \frac{1}{a-i\alpha} = \frac{2a}{a^2+\alpha^2} \quad (4)$$

(b) $$-\alpha^2 U = \frac{1}{k}\frac{\partial U}{\partial t} \qquad U = e^{-k\alpha^2 t} \cdot C \quad (4+2)$$

$$U(\alpha, 0) = \frac{2a}{a^2+\alpha^2} \Rightarrow C = \frac{2a}{a^2+\alpha^2} \quad (4)$$

$$u = \mathcal{F}^{-1}\{U\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2a}{a^2+\alpha^2} e^{-k\alpha^2 t} \cdot e^{-i\alpha x} dx. \quad (4)$$

2. Solve the boundary value problem using seperation of variables

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < 1, \quad 0 < y < 2$$

$$u(0, y) = 0, \quad u(1, y) = 0,$$

$$u(x, 0) = 0, \quad u(x, 2) = x.$$

$U = X \cdot Y \qquad X''Y + XY'' = 0 \qquad \frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$

$X'' + \lambda X = 0 \quad X(0) = 0 \quad X(1) = 0$ (4+2)

(i) $\lambda = 0 \Rightarrow X = c_1 + x c_2 \Rightarrow c_1 = c_2 = 0$ trivial solution 2

(ii) $\lambda < 0 \Rightarrow \lambda = -\alpha^2 \quad X = c_1 \cosh \alpha x + c_2 \sinh \alpha x \Rightarrow c_1 = 0$ from $X(0) = 0$

$X(1) = c_2 \sinh \alpha = 0 \Rightarrow c_2 = 0$ trivial solution.

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(iii) $\lambda = \alpha^2 > 0 \qquad X = c_1 \sin \alpha x + c_2 \cos \alpha x$

$X(0) = c_2 = 0 \qquad X(1) = c_1 \sin \alpha = 0$

$\alpha = n\pi \qquad \lambda = n^2\pi^2$

$X = c_1 \sin n\pi x$

$Y'' - \lambda Y = 0 \quad Y'' - n^2\pi^2 Y = 0 \qquad Y = c_1 \sinh n\pi y + c_2 \cosh n\pi y$ 4

$Y(0) = 0 \Rightarrow c_2 = 0$

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$u_n = X_n Y_n = A_n \sinh n\pi y \sin n\pi x$

$u = \sum_{n=1}^{\infty} A_n \sinh n\pi y \sin n\pi x \qquad u(x,2) = x = \sum_{n=1}^{\infty} A_n \sinh 2n\pi \sin n\pi x$ 2

$\sinh 2n\pi\, A_n = 2 \int_0^1 x \sin n\pi x \, dx = 2\left(-x \frac{\cos n\pi x}{n\pi}\Big|_0^1 + \int_0^1 \frac{\cos n\pi x}{n\pi} dx\right)$

$= +2\left(\frac{(-1)^{n+1}}{n\pi} + \frac{\sin n\pi x}{(n\pi)^2}\Big|_0^1\right) = \frac{2(-1)^{n+1}}{n\pi}.$

$$\boxed{\sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \frac{\sinh n\pi y}{\sinh 2n\pi} \sin n\pi x}$$

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3. Use Laplace transform to solve the problem

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{9}\frac{\partial^2 u}{\partial t^2}, \quad 0 < x < 1, \quad t > 0$$

$$u(0,t) = 0, \quad u(1,t) = 0, \quad t > 0,$$

$$u(x,0) = 0, \quad u_t(x,0) = \sin 2\pi x$$

$$9\frac{\partial^2 U}{\partial x^2} = s^2 U(x,s) - s\underbrace{u(x,0)}_{=0} - \underbrace{u_t(x,0)}_{=\sin 2\pi x}.$$

(4+2)

$$\frac{\partial^2 U}{\partial x^2} - \frac{s^2}{9}U = -\frac{\sin 2\pi x}{9}$$

$$U_c = c_1\cosh\frac{s}{3}x + c_2\sinh\frac{s}{3}x \qquad U_p = A\sin 2\pi x + B\cos 2\pi x$$ (4+2)

$$-4\pi^2 A\sin 2\pi x - 4\pi^2 B\cos 2\pi x - \frac{As^2}{9}\sin 2\pi x - \frac{Bs^2}{9}\cos 2\pi x = -\frac{\sin 2\pi x}{9}$$

$$B = 0 \qquad A = \frac{1}{36\pi^2 + \frac{s^2}{9}} = \frac{1}{s^2 + 36\pi^2}$$ (4)

$$U = c_1\cosh\frac{s}{3}x + c_2\sinh\frac{s}{3}x + \frac{1}{s^2+36\pi^2}\sin 2\pi x$$ (2)

$$U(0,s) = 0 \Rightarrow c_1 = 0$$

$$U(1,s) = 0 \Rightarrow c_2\sinh\frac{s}{3} = 0 \Rightarrow c_2 = 0$$ (2)

$$U = \frac{1}{s^2+36\pi^2}\sin 2\pi x \qquad u(x,t) = \frac{1}{6\pi}\sin 6\pi t\,\sin 2\pi x$$ (4)

4. The diplacement $u(r,t)$ of a circular membrane of radius $c = 2$ clamped along its circumference with initial displacement and initial velocity in polar coordinates modelled by

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = \frac{\partial^2 u}{\partial t^2}, \quad 0 < r < 2, \quad 0 < t$$

$$u(2,t) = 0, \quad t > 0,$$

$$u(r,0) = 0, \quad u_t(r,0) = 2.$$

Using seperation of variables find $u(r,t)$.

$u = R \cdot T \qquad R''T + \frac{1}{r}R'T = RT'' \Rightarrow \dfrac{R'' + \frac{1}{r}R}{R} = \dfrac{T''}{T} = -\lambda$ (4+2)

$r^2 R'' + r R' + \lambda r^2 R = 0$ (Bessel equation of order $n=0$)

$\lambda = \alpha^2 \qquad R = c_1 J_0(\alpha r) + c_2 Y_0(\alpha r)$ (2)

Since R is bounded, $c_2 = 0$. (2)

$U(2,t) = 0 \quad R(2) = 0$. Hence, $J_0(2\alpha) = 0$

$\lambda_n = \alpha_n^2$ satisfying $J_0(2\alpha_n) = 0 \qquad R_n = J_0(\alpha_n r)$ (2)

$T'' + \lambda T = 0 \quad \lambda = \alpha_n^2 \quad T = c_1 \cos \alpha_n t + c_2 \sin \alpha_n t$

$u(r,0) = 0 \quad T(0) = 0 \Rightarrow c_1 = 0$ (4)

$$u(r,t) = \sum_{n=1}^{\infty} A_n \sin \alpha_n t \, J_0(\alpha_n r) \quad (2)$$

$$u_t(r,0) = 2 \qquad 2 = \sum_{n=1}^{\infty} A_n \alpha_n J_0(\alpha_n r)$$

$t = \alpha_n x \qquad dt = \alpha_n dx$

$$\alpha_n A_n = \frac{2 \cdot 2}{2^2 J_1^2(2\alpha_n)} \int_0^2 x J_0(\alpha_n x)\, dx$$

$$A_n = \frac{1}{\alpha_n^3 J_1^2(2\alpha_n)} \cdot \int_0^{2\alpha_n} \frac{d(t J_1 t)}{dt}\, dt = \frac{2\alpha_n J_1(2\alpha_n)}{\alpha_n^3 J_1^2(2\alpha_n)} = \frac{2}{\alpha_n^2 J_1(2\alpha_n)}$$

(2+4) = (6)

5. Find the first three non-zero terms in the Fourier-Legendre expansion of

$$f(x) = \begin{cases} x & -1 < x < 0 \\ 1 & 0 \le x < 1 \end{cases}$$

$$c_0 = \frac{1}{2}\int_{-1}^{1} f(x)P_0(x)\,dx = \frac{1}{2}\int_{-1}^{0} x\cdot 1\,dx + \frac{1}{2}\int_{0}^{1} 1\cdot 1\,dx$$

$$= \frac{x^2}{4}\Big|_{-1}^{0} + \frac{x}{2}\Big|_{0}^{1} = -\frac{1}{4} + \frac{1}{2} = \frac{1}{4} \qquad (4)$$

$$c_1 = \frac{3}{2}\int_{-1}^{1} f(x)P_1(x)\,dx = \frac{3}{2}\int_{-1}^{0} x\cdot x\,dx + \frac{3}{2}\int_{0}^{1} 1\cdot x\,dx$$

$$= \frac{1}{2}x^3\Big|_{-1}^{0} + \frac{3}{4}x^2\Big|_{0}^{1} = \frac{1}{2} + \frac{3}{4} = \frac{5}{4} \qquad (6)$$

$$c_2 = \frac{5}{2}\int_{-1}^{1} f(x)P_2(x)\,dx = \frac{5}{2}\int_{-1}^{0} x\,\frac{1}{2}(3x^2-1)\,dx + \frac{5}{2}\int_{0}^{1}\frac{1}{2}(3x^2-1)\,dx$$

$$= \frac{5}{2}\left(\frac{3}{2}\cdot\frac{x^4}{4} - \frac{x^2}{4}\right)\Big|_{-1}^{0} + \frac{5}{2}\left(\frac{1}{2}x^3 - \frac{x}{2}\right)\Big|_{0}^{1}$$

$$= -\frac{15}{16} + \frac{5}{8} + \frac{5}{4} - \frac{5}{4} = -\frac{5}{16} \qquad (6)$$

$$f(x) = \frac{1}{4}P_0 + \frac{5}{4}P_1 - \frac{5}{16}P_2 + \cdots \qquad (4)$$

6. Find the steady-state temperature $u(r,\theta)$ in a sphere of unit radius by solving the problem

$$\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta}=0,\quad 0<r<1,\quad 0<\theta<\pi,$$

$$u(1,\theta)=\cos^2\theta,\quad 0<\theta<\pi.$$

$u = R\cdot\Theta$

$$R''\Theta+\frac{2}{r}R'\Theta+\frac{1}{r^2}R\Theta''+\frac{\cot\theta}{r^2}R\Theta'=0$$

$$\frac{R''+\frac{2}{r}R'}{R}=-\frac{1}{r^2}\frac{\Theta''+\cot\theta\,\Theta'}{\Theta}$$

$$\frac{r^2R''+2rR'}{R}=-\frac{\Theta''+\cot\theta\,\Theta'}{\Theta}=\lambda \qquad (4)$$

$\Theta''+\cot\theta\,\Theta'+\lambda\Theta=0 \qquad x=\cos\theta$

$\Rightarrow (1-x^2)\Theta''(x)-2x\Theta'(x)+\lambda\Theta(x)=0 \qquad \lambda=n(n+1)\quad n=0,1,\ldots$

$\Theta_n(x)=P_n(x) \qquad \Theta_n(\theta)=P_n(\cos\theta) \qquad (8)$

$r^2R''+2rR'-\lambda R=0 \quad \lambda_n=n(n+1) \quad$ C-E equ.

$m^2+m-n(n+1)=0 \qquad m_{1,2}=n,\,-(n+1)$

$R=c_1r^n+c_2r^{-(n+1)} \qquad u(r,\theta)$ is bounded $\Rightarrow c_2=0. \qquad (6)$

$u(r,\theta)=\sum_{n=0}^{\infty}A_nr^nP_n(\cos\theta). \qquad u(1,\theta)=\cos^2\theta=\sum_{n=0}^{\infty}A_nP_n(\cos\theta)$

~~$\cos^2\theta=$~~ $\quad A_n=\frac{2n+1}{2}\int_0^\pi \cos^2\theta\,P_n(\cos\theta)\sin\theta\,d\theta$

$\cos^2\theta=\frac{2}{3}\left(\frac{1}{2}(3\cos^2\theta-1)\right)+\frac{1}{3}=\frac{2}{3}P_2(\cos\theta)+\frac{1}{3}P_0(\cos\theta)$

By orthogonality $\quad A_n=0$ if $n\neq 0,2$

$=0 \quad A_0=\frac{1}{2}\int_0^\pi \frac{1}{3}P_0^2(\cos\theta)\sin\theta\,d\theta=\frac{1}{6}(-\cos\theta)\Big|_0^\pi=\frac{1}{3}.$ $\Rightarrow$

$$A_2 = \frac{5}{2} \int_0^{\pi} \frac{2}{3} P_2^2(\cos\theta)\sin\theta\, d\theta$$

$$= \frac{5}{3} \int_{-1}^{1} P_2^2(x)\, dx = \frac{5}{3} \int_{-1}^{1} \frac{1}{4}(3x^2-1)^2\, dx$$

$$= \frac{5}{3\cdot 4} \int_{-1}^{1} 9x^4 - 6x^2 + 1\, dx$$

$$= \frac{5}{12}\left(\frac{9x^5}{5} - 2x^3 + x \right)\Bigg|_{-1}^{1}$$

$$= \frac{5}{12} \cdot 2 \cdot \left(\frac{9}{5} - 2 + 1 \right) = \frac{5}{6}\,\frac{4}{5} = \frac{2}{3}. \qquad (4)$$

$$u(r,\theta) = \frac{1}{3} + \frac{2}{3} P_2(\cos\theta) r^2$$

$$= \frac{1}{3} + \frac{2}{3}\left(\frac{1}{2}(\cos^2\theta - 1) r^2 \right) \qquad (2)$$