1. Evaluate

5 (a) (-points) $\mathcal{L}\{te^{-t}\cos^2 2t\} = \mathcal{L}\left\{t e^{-t}\, \frac{1+\cos 4t}{2}\right\}$

✓

$$= \frac{1}{2}\mathcal{L}\{t e^{-t}\} + \frac{1}{2}\mathcal{L}\{t e^{-t}\cos 4t\}$$

$$= \frac{1}{2}\frac{1}{(s+1)^2} + \frac{1}{2}\mathcal{L}\{t\cos 4t\}_{s\to s+1} = \frac{1}{2}\frac{1}{(s+1)^2} + \frac{1}{2}\frac{d}{ds}\left(\frac{s}{s^2+16}\right)_{s\to s+1}$$

$$= \frac{1}{2}\frac{1}{(s+1)^2} + \frac{1}{2}\frac{-(s+1)^2+16}{((s+1)^2+16)^2}$$

5 (b) (-points) $\mathcal{L}\{f(t)\}$, where $f(t) = \begin{cases} t & 0 \le t \le 1 \\ 1 & 1 \le t \end{cases}$.

✓

$$f(t) = t(1-u(t-1)) + u(t-1) = t - (t-1)u(t-1)$$

$$\mathcal{L}\{f(t)\} = \frac{1}{s^2} - \frac{1}{s^2}e^{-s}$$

5 (c) (-points) $\mathcal{L}\{f(t)\}$, where $f(t) = 1 - e^{-t}$, for $0 < t < 2$ and $f(t+2) = f(t)$.

✓

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-2s}}\int_0^2 (1-e^{-t})\, e^{-st}\, dt$$

$$= \frac{1}{1-e^{-2s}}\left(\frac{e^{-st}}{-s} + \frac{e^{-t(1+s)}}{1+s}\right)\Bigg|_0^2$$

$$= \frac{1}{1-e^{-2s}}\left(\frac{1-e^{-2s}}{s} - \frac{1-e^{-2(s+1)}}{1+s}\right)$$

a) $\mathcal{L}^{-1}\left\{\dfrac{1}{(s^2+4)^2}\right\}$ 5

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\cdot\frac{1}{s^2+4}\right\} = \frac{1}{2}\sin 2t * \sin 2t$$

$$= \frac{1}{2}\int_0^t \sin 2\tau \sin 2(t-\tau)\, d\tau = \frac{1}{4}\int_0^t \frac{1}{2}(\cos(4\tau-2t) - \cos 2t\, d\tau$$

$$= \frac{1}{8}\left(\frac{\sin(4\tau-2t)}{4} - \tau\cos 2t\right)_{\tau=0}^{t}$$

$$= \frac{1}{16}\sin 2t - \frac{1}{8}t\cos 2t$$

b) $\mathcal{L}^{-1}\left\{\dfrac{e^{-2s}}{s^2-1}\right\}$ 5

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2-1}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-1} - \frac{1}{s+1}\right\}$$

$$= \frac{1}{2}e^{t} - \frac{1}{2}e^{-t}$$

$$\mathcal{L}\left\{\frac{e^{-2s}}{s^2-1}\right\} = \left(\frac{1}{2}e^{t-2} - \frac{1}{2}e^{t-2}\right)u(t-2).$$

c) $$\mathcal{L}^{-1}\left\{\frac{s}{s^2+4s+5}\right\} = \mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+1}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+1}\right\}$$

$$= \cos t\, e^{-2t} - 2\sin t\, e^{-2t}$$ 5

3. (-points) Solve the initial value problem using Laplace transform.

10

$$y'' + 6y' + 5y = \delta(t-2)$$

$$y(0) = 1, \quad y'(0) = 0.$$

$$s^2 Y(s) - s + 6s\,Y(s) - 6 + 5Y(s) = e^{-2s}$$

$$(s^2+6s+5)\,Y(s) = e^{-2s} + s + 6$$

3

$$Y(s) = \frac{e^{-2s}}{(s+5)(s+1)} + \frac{s+6}{(s+5)(s+1)}$$

1

$$y(t) = \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s+1} - \frac{e^{-2s}}{s+5}\right\} + \frac{5}{4}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s+5}\right\}$$

$$= \frac{1}{4}e^{-(t-2)}u(t-2) - \frac{1}{4}e^{-5(t-2)}u(t-2) + \frac{5}{4}e^{-t} - \frac{1}{4}e^{-5t}$$

2 + 2 2

for the shift

10

$$y + 5\int_0^t \cos 2(t-\tau)\, y(\tau)\, d\tau = 10$$ [illegible]

$$Y(s) + 5\,\frac{s}{s^2+4}\,Y(s) = \frac{10}{s}$$

5

$$\frac{s^2+5s+4}{s^2+4}\,Y(s) = \frac{10}{s}$$

$$Y(s) = \frac{10(s^2+4)}{s(s+4)(s+1)} = \frac{-50}{3(s+1)} + \frac{50}{3(s+4)} + \frac{10}{s}$$

$$y(t) = -\frac{50}{3}e^{-t} + \frac{50}{3}e^{-4t} + 10$$

5

5

5. (-points) Show that the set $\{2x+3, 45x^2+9x-17\}$ is an orhogonal set on $[-1,1]$. Find the norm of $2x+3$.

$$\int_{-1}^{1} (2x+3)(45x^2+9x-17)\,dx$$

$$= \int_{-1}^{1} \underbrace{90x^3}_{\text{odd}} + 18x^2 - \underbrace{34x}_{\text{odd}} + 135x^2 + \underbrace{27x}_{\text{odd}} - 51 \, dx$$

$$= 51x^3 - 51x \Big|_{-1}^{1} = 0$$

2

$$\|2x+3\| = \sqrt{\int_{-1}^{1} (2x+3)^2\,dx} = \sqrt{\frac{4x^3}{3} + 6x^2 + 9x \Big|_{-1}^{1}}$$

$$= \sqrt{\frac{62}{3}}$$

2

(a) Find the Fourier series of $f(x) = |x|$ $-\pi < x < \pi$.

(b) Find the sum $1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots$

using part (a).

(a) $|x|$ is an even function. $\Rightarrow b_n = 0$. (1)

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \cos nx \, dx = \frac{2}{\pi}\int_0^{\pi} x\cos(nx)\,dx$$

$$= \frac{2}{\pi}\left(\left(\frac{x\sin(nx)}{n}\right)_0^{\pi} - \int_0^{\pi}\frac{\sin(nx)}{n}\,dx\right) = \frac{2}{\pi}\left(\frac{\cos nx}{n^2}\right)_0^{\pi}$$

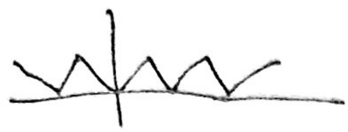
$$= \frac{2((-1)^n - 1)}{\pi n^2} \quad (5)$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\,dx = \pi \quad (2)$$

$$|x| = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2}\cos(nx)$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \dots\right) \quad (2)$$

(b) $x = \pi$ (point of continuity)



$$\pi = \frac{\pi}{2} - \frac{4}{\pi}\left(-1 - \frac{1}{9} - \frac{1}{25} - \dots\right)$$

$$1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \dots = \frac{\pi^2}{8} \quad (5)$$

6

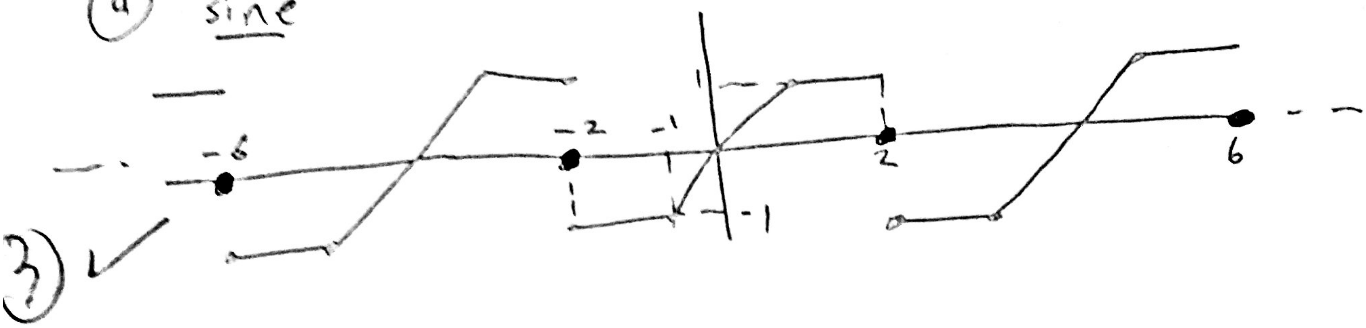
7\. a) (-points) Sketch the periodic extension to which the half-range sine, cosine and ~~Fourier~~ ~~expansion~~ of $f(x)$ converges, where

$$f(x) = \begin{cases} x & 0 < x < 1 \\ 1 & 1 \leq x < 2 \end{cases}$$

4

b) (-points) Determine the values to which the half-range sine, cosine and ~~Fourier series of~~ $f(x)$ converges at $x = -1$ and $x = -6$.

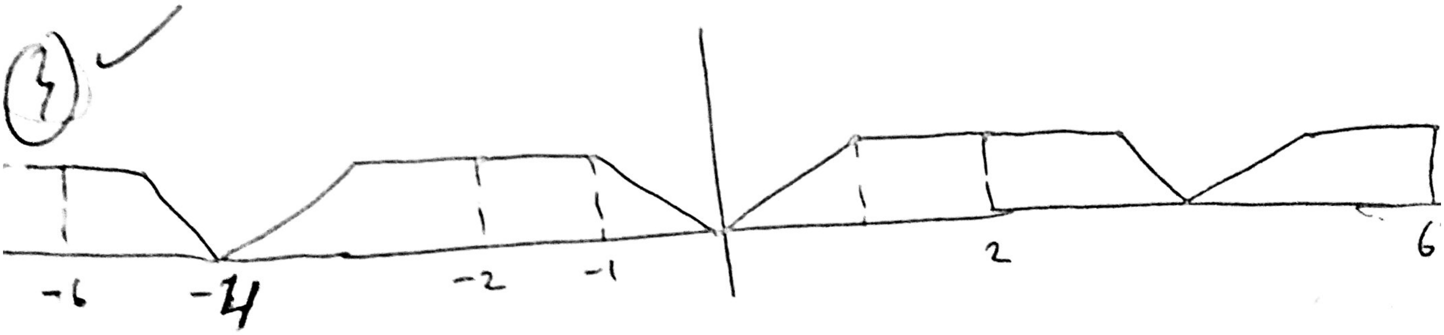
(a) sine



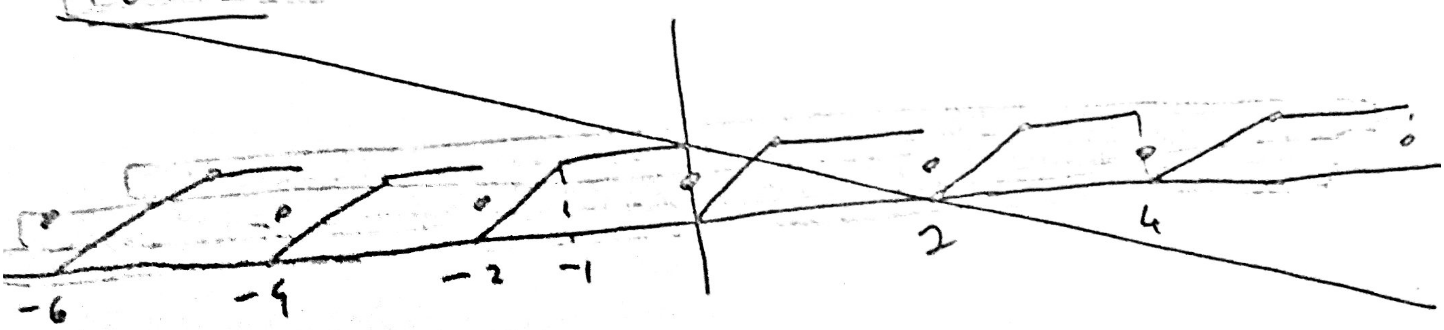
3

cosine

3



~~Fourier~~



	sine	cosine	~~Fourier~~
-1	-1	1	~~1~~
-6	0	1	~~1/2~~

nd eigenvalues and eigenfunctions of S-L problem

$y'' + 2y' + \lambda y = 0 \quad y(0) = 0 \quad y(1) = 0$

(b) Convert the equation to the self-adjoint form and find its weight function.

(c) Write the orthogonality relation.

) $m^2 + 2m + \lambda = 0 \qquad m_{1,2} = -1 \mp \sqrt{1-\lambda}$

case 1 $\quad 1-\lambda = 0 \qquad m_{1,2} = -1 \qquad y = c_1 e^{-x} + c_2 x e^{-x}$

$y(0) = 0 \qquad c_1 = 0 \qquad y = c_2 x e^{-x}$ (2)

$y(1) = 0 \qquad c_2 e^{-1} = 0 \Rightarrow c_2 = 0 \qquad \boxed{y=0}$ trivial solution.

ase 2 $\quad 1-\lambda > 0 \qquad 1-\lambda = \alpha^2 \qquad m_{1,2} = -1 \mp \alpha$

$y = c_1 e^{(-1+\alpha)x} + c_2 e^{(-1-\alpha)x}$

$y(0) = 0 \Rightarrow c_1 + c_2 = 0 \qquad c_2 = -c_1 \qquad y = c_1 e^{(-1+\alpha)x} - c_1 e^{(-1-\alpha)x}$

$y(1) = 0 \qquad y(1) = c_1 \underbrace{\left(e^{-1+\alpha} - e^{-1-\alpha}\right)}_{\neq 0} = 0 \qquad c_1 = 0$ (4)

$\Rightarrow \boxed{y=0}$ trivial solution

se 3 $\quad 1-\lambda < 0 \qquad 1-\lambda = -\alpha^2 \qquad m = -1 \mp \alpha i$

$y = e^{-x}(c_1 \cos \alpha x + c_2 \sin \alpha x)$ (2)

$y(0) = 0 \Rightarrow \quad c_1 = 0 \qquad y = c_2 e^{-x} \sin \alpha x$

$y(1) = 0 \Rightarrow \quad c_2 e^{-2} \sin \alpha = 0 \Rightarrow \sin \alpha = 0$

$\alpha = n\pi \qquad 1 - \lambda = -n^2\pi^2 \qquad \lambda_n = 1 + n^2\pi^2$ (eigenvalues) (2)

$y_n = e^{-x} \sin n\pi x$ are eigenfunctions (2)

$r(x) = e^{\int 2\,dx} = e^{2x}$

6

$\frac{d}{dx}\left(e^{2x} y'\right) + \lambda e^{2x} y = 0 \qquad p(x) = e^{2x}$

)

$$\int_0^1 e^{2x}\left(e^{-x}\sin(n\pi x)\right)e^{-x}\sin(m\pi x)\,dx = 0$$

2

$n \neq m$

(-1) if not written