Q1. (a) Find the length of the curve traced by $r(t) = 12t\, i + 8t^{3/2} j + 3t^2 k$ on the interval $0 \le t \le 2$.

(b) Find the equation of tangent line to the curve at $t = 1$.

(10 pts)

(a) $\ell = \int_0^2 \sqrt{12^2 + (12t^{1/2})^2 + (6t)^2}\, dt = \int_0^2 \sqrt{144 + 144t + 36t^2}\, dt$ 2

$= \int_0^2 6\sqrt{4 + 4t + t^2}\, dt$

$= 6 \int_0^2 (t+2)\, dt = 6\left(\frac{t^2}{2} + 2t\right)\Big|_0^2$ 2

$= 6(2+4) = 36$ 2

(b) $r'(1) = \langle 12, 12, 6 \rangle$ 1

$\ell(t) = \langle 12, 12, 6 \rangle t + \underbrace{\langle 12, 8, 3 \rangle}_{r(1)}$ 1

$x = 12t + 12$ 2

$y = 12t + 8$ $\quad -\infty \le t \le \infty$

$z = 6t + 3$

Q2.

(a) Find the directional derivative of the $f(x, y, z) = xy^2 \tan^{-1} z$ at (2,1,1) in the direction of $i + j + k$. 6

(b) Find the direction in which f decreases most rapidly and the value of maximum rate of change at (−2,3,1). 4

(10 pts)

(a) $D_u f = \nabla f \cdot u$

$\nabla f = \left\langle y^2 \tan^{-1} z,\ 2xy \tan^{-1} z,\ \frac{xy^2}{1+z^2} \right\rangle$ 2

$\nabla f_{(2,1,1)} = \left\langle \tan^{-1} 1,\ 4 \tan^{-1} 1,\ \frac{2}{1+1^2} \right\rangle = \left\langle \pi/4,\ \pi,\ 1 \right\rangle$

$u = \frac{\langle 1, 1, 1 \rangle}{\sqrt{1^2 + 1^2 + 1^2}} = \left\langle 1/\sqrt{3},\ 1/\sqrt{3},\ 1/\sqrt{3} \right\rangle$ 2

$\nabla f \cdot u = \left\langle \frac{\pi}{4\sqrt{3}} + \frac{\pi}{\sqrt{3}} + \frac{1}{\sqrt{3}} \right\rangle$ 2

(b) The direction f decreases most rapidly is

$-\nabla f = \left\langle 9 \tan^{-1} 1,\ -12 \tan^{-1} 1,\ \frac{-2 \cdot 3^2}{1 + 1^2} \right\rangle$

$= \left\langle -\frac{9\pi}{4},\ 3\pi,\ 9 \right\rangle$ 2

The maximum rate of change is $|\nabla f| = \sqrt{\left(\frac{9\pi}{4}\right)^2 + (3\pi)^2 + 9^2}$

$= \sqrt{\frac{129}{16}\pi + 81}$ 2

Q3. Evaluate $\nabla \cdot (\nabla \times F)$ where $F(x, y, z) = x^2\,i + y^2\,j - 2xz\,k$.

(5 pts)

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 & y^2 & -2xz \end{vmatrix} \qquad 2$$

$$= 0i - (-2z - 0)j + (0 - 0)k = 2zj$$

$$\nabla \cdot (2zj) = \frac{\partial}{\partial y}(2z) = 0. \qquad 3$$

Q4. Show that $F(x, y, z) = 2xyz\,i + x^2z\,j + x^2y\,k$ is conservative. Evaluate $\int_{(2,1,-1)}^{(5,1,2)} F \cdot dr$ using a potential φ of F.

(15 pts)

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xyz & x^2z & x^2y \end{vmatrix}$$

$$= (x^2 - x^2)i - (2xy - 2xy)j + (2xz - 2xz)k = \vec{0}. \qquad 4$$

$\Rightarrow$ F is conservative.

$$\frac{\partial \varphi}{\partial x} = 2xyz \qquad \frac{\partial \varphi}{\partial y} = x^2z \qquad \frac{\partial \varphi}{\partial z} = x^2y \qquad 2$$

$$\varphi = x^2yz + g(y, z) \qquad 2$$

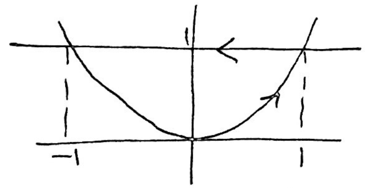
$$\frac{\partial \varphi}{\partial y} = x^2z + \frac{\partial g}{\partial y} = x^2z \qquad \frac{\partial g}{\partial y} = 0 \quad \Rightarrow g(y, z) = f(z) \qquad 2$$

$$\frac{\partial \varphi}{\partial z} = x^2y + f'(z) = x^2y \qquad f(z) = c \qquad \varphi = x^2yz + c \qquad 2$$

$$\int_{(2,1,-1)}^{(5,1,2)} F \cdot dr = \varphi(5,1,2) - \varphi(2,1,-1) = 50 - (-4) = 54. \qquad 3$$

Q5. Use Green's theorem to evaluate the integral $\int_C (e^{x^2} + xy^2)dx + (\ln y - x^2 y)dy$, where the closed path C consists of the parabola $y = x^2$ and the line $y = 1$, oriented counterclockwise.

(15pts)



(2)

$$\oint_C F\cdot dr = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

2+2

$$= \int_{-1}^{1}\int_{x^2}^{1} -2xy - 2xy \; dy\, dx \quad 2+1$$

(4+3)

$$= \int_{-1}^{1} -2xy^2 \Big|_{y=x^2}^{y=1} dx = \int_{-1}^{1} -2x(1-x^4)\,dx$$

(4)

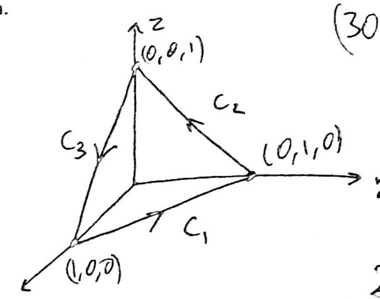
$$= -x^2 + 2\frac{x^6}{6}\Big|_{-1}^{1}$$

$$= -1 + \frac{1}{3} - \left(-1 + \frac{1}{3}\right) = 0$$

(2)

Q6. Let S be the part of the plane $x + y + z = 1$ in the first octant, oriented upward. If $F(x,y,z) = (x^2 - y^2)\,i + (y^2 - z^2)\,j + (z^2 - x^2)\,k$ verify Stokes' Theorem by calculating both integrals in the statement of the theorem.

(30pts)



2

$$\oint_C F\cdot dr = \int_{C_1} + \int_{C_2} + \int_{C_3}$$

2

C_1:

$r(t) = \langle 1,0,0\rangle + t\,\langle 0-1, 1-0, 0-0\rangle$

$= \langle 1-t, t, 0\rangle \quad 0 \le t \le 1$

$$\int_{C_1} \langle 1-2t, t^2, -(1-t)^2\rangle \cdot \underbrace{\langle -1,1,0\rangle dt}_{dr}$$

4

$$= \int_0^1 (t^2+2t-1)\,dt = \frac{1}{3}$$

2:

$r(t) = \langle 0, 1-t, t\rangle \quad 0 \le t \le 1$

$$\int_{C_2} \langle -(1-t)^2, 1-2t, t^2\rangle \cdot \langle 0,-1,1\rangle dt = \int_0^1 (t^2+2t-1)\,dt = \frac{1}{3}$$

4

$r(t) = \langle t, 0, 1-t\rangle \Rightarrow \int_{C_3} = \int_0^1 (t^2+2t-1)\,dt = \frac{1}{3}$. 4

$$\oint_C = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1.$$

2

S

$z = 1 - x - y$

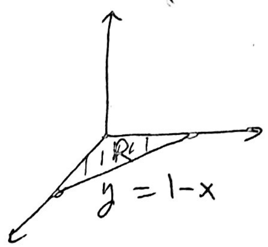
$$\nabla \times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2-y^2 & y^2-z^2 & z^2-x^2 \end{vmatrix} = \langle 2z, 2x, 2y \rangle$$ 3

$$n = \frac{\nabla g}{\|\nabla g\|} = \langle \tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}} \rangle \qquad dS = \sqrt{1^2+1^2+1^2}\, dA = \sqrt{3}\, dA$$ 3

$$\iint_R \langle 2(1-x-y), 2x, 2y \rangle \langle 1, 1, 1 \rangle \, dA$$

$$= \iint_R 2\, dy\, dx = \int_0^1 \int_0^{1-x} 2\, dy\, dx = 1$$

(4+2)



Q7. Let E be the solid cube defined by $0 \le x \le 1, 0 \le y \le 1$ and $0 \le z \le 1$. Let S be the boundary of E. If $F(x,y,z) = 2xy\ i + 3ye^z\ j + x\sin z\ k$, find $\iint_S (F \cdot n) dS$ using the divergence theorem. (15 pts)

$$\text{div} F = \frac{\partial}{\partial x}(2xy) + \frac{\partial}{\partial y}(3ye^z) + \frac{\partial}{\partial z}(x\sin z)$$ 5

$$= 2y + 3e^z + x\cos z$$

$$\iint_S (F \cdot n) dS = \iiint_E \text{div} F\, dV = \int_0^1 \int_0^1 \int_0^1 (2y + 3e^z + x\cos z)\, dx\, dy\, dz$$ 5

$$= \int_0^1 \int_0^1 (2y + 3e^z + \tfrac{1}{2}\cos z)\, dy\, dz$$

$$= \int_0^1 (1 + 3e^z + \tfrac{1}{2}\cos z)\, dz$$ 3

$$= 1 + 3(e-1) + \tfrac{1}{2}\sin 1.$$

2