Department of Mathematics and Statistics, KFUPM
Math 280, Term 172
Final Exam , May 09, 2018, Duration: 210 minutes

Name: ID:

Exercise 1(20points, 5-5-5-5).
Consider a non-homogeneous system (S): $AX = Y$, where A is an $n \times n$ matrix.
(1) Prove that if A is invertible, then the system has exactly one solution.
(2) Assume that X_1, X_2 are two solutions of (S) and α_1, α_2 real numbers. Under which condition $\alpha_1 X_1 + \alpha_2 X_2$ is a solution of (S).
(3) Verify that $X_1 = (-1, 0, 1)$ and $X_2 = (0, -1, 0)$ are solutions of the system (S)

$$\begin{pmatrix} 2x & + & y & + & z & = & -1 \\ x & - & y & + & 2z & = & 1 \\ x & + & 2y & - & z & = & -2 \end{pmatrix}$$

(4) Without any calculations, justify why $X_1 + X_2$ is not a solution while $\frac{1}{3}X_1 + \frac{2}{3}X_2$ is a solution of (S). [**Do not solve the system**].

Exercise 2(20points, 5-5-5-5).
Let $V = \mathbb{R}^3$ endowed with the standard inner product
$W = \{(a, b, c) \in \mathbb{R}^3 | a + b - 2c = 0\}$ and $F = \{(a, b, c) \in \mathbb{R}^3 | a - b - 2c = 0\}$.
(1) Which one of the subsets W, F is a subspace of V.
(2) Is $W \cup F$ a subspace of V?
(3) Are W and F orthogonal?
(4) Find the orthogonal subspace W^T of W.

Exercise 3(20points, 5-5-5-5).
Let $V = \mathcal{M}_2(\mathbb{R})$ be the vector space of 2×2 real matrices, A a 2×2 fixed matrix and $T : V \to V$ defined by $T(B) = AB$ for every $B \in V$.
(1) Prove that T is a linear transformation.
(2) Prove that if A is invertible, then T has an inverse, that is, $T^{-1} : V \longrightarrow V$ such that $ToT^{-1} = T^{-1}oT = id$ where $id : V \longrightarrow V$ is the identity map.
(3) Put $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ and let $S = \{M_1, M_2, M_3, M_4\}$ be the standard basis of V. Find $[T]_S$.
(4) Find $[T^{-1}]_S$.

Exercise 4(20points, 5-5-5-5).
Let $V = \mathbb{R}^3$, $S = \{e_1, e_2, e_3\}$ the standard basis of V and $T : V \to V$ defined by $T(a, b, c) = (a + b + 2c, b, b + 3c)$.
(1) Find the matrix $A = [T]_S$ representing T is the basis S.
(2) Prove that A is diagonalizable.
(3) Find a matrix P such that $P^{-1}AP$ is diagonal.
(4) Find a basis B of V such that the matrix $[T]_B$ is diagonal.

Exercise 5 (20points, 5-5-5-5).
Let A be a 3×3 matrix and $f(X) = X^3 + a_2X^2 + a_1X + a_0$ its characteristic polynomial.
(1) Prove that if $a_0 \neq 0$, then A is invertible and find its inverse.
(2) If $a_0 = 0$, is A necessarily invertible? Justify by an example.
(3) Suppose that A is diagonalizable and D its diagonal matrix is invertible. Prove that A is invertible.
(4) Assume that all eigenvalues of A are equal to 1 or -1. Prove that $A = A^{-1}$.

Exercise 6 (20points, 5-5-5-5).
Let $V = \mathcal{C}([0,1])$ be the vector space of all continuous functions on $[0,1]$ endowed with the inner product $(f|g) = \int_0^1 f(x)g(x)dx$.
(1) Determine the angle θ between 1 and x.
(2) Find the the vector projection p of 1 onto x.
(3) Verify that $1 - p$ is orthogonal to p.
(4) Compute $\|1 - p\|$, $\|p\|$ and $\|1\|$ and verify Pythagore's law.

Exercise 7 (20points, 4-7-9).
Let $p_0(x), p_1(x)$ and $p_2(x)$ orthogonal with respect to the inner product $(p(x)|q(x)) = \int_0^1 \frac{p(x)q(x)}{1+x^2}dx$. Find $p_0(X), p_1(x)$ and p_2 if all polynomials have leading coefficient equal to 1.

Exercise 8(20points, 6-8-3-3).
Consider the quadratic form q of $\mathbb{R}^3$ defined by $q(x, y, z) = 3x^2 - 3z^2 + 8yz$.
(1) Write q in the matrix form and find the eigenvalues of its matrix.
(2) Find a the canonical quadratic form associated to q.
(3) Find the signature of q.
(4) Find the rank of q.

Exercise 9(20points, 5-5-5-5).
Consider the quadratic equation $5x^2 + 5y^2 - 6xy - 24\sqrt{2}x + 8\sqrt{2}y + 56 = 0$.
(1) Write the equation in the matrix form and find the eigenvalues and eigenvectors of its matrix A.
(2) Find an orthogonal matrix P and use the substitution $X = PX'$ to transform the equation to a simple form.
(3) Identify the obtained new equation with its rotation/translation axes.
(4) Sketch the graph of the equation.

Exercise 10(20points, 6-8-6).
Consider the function $F(x, y) = 3x^2 - xy + y^2$.
(1) Find all stationary points of $F(x, y)$.
(2) Find the Hessian matrix $H(X_0)$ and its eigenvalues for each stationary point X_0.
(3) Classify the stationary points (local max, local min, and saddle)