Department of Mathematics and Statistics, KFUPM
Math 280, Term 172
Exam 3, April 08, 2018, Duration: 120 minutes

Name: ID:

Exercise 1(10points, 5-5).
Let V be a vector space of finite dimension n, $T : V \mapsto V$ a linear transformation on V and B and S bases for V.
(1) Prove that $[T]_{BB}$ and $[T]_{SS}$ are similar.
(2) Assume that $n = 3$ and let $P = \begin{pmatrix} 2 & 3 & 2 \\ 2 & -1 & 4 \\ 3 & 2 & 0 \end{pmatrix}$, the transition matrix from B to S and $[T]_{BB} = \begin{pmatrix} 2 & 3 & 2 \\ 2 & -1 & 4 \\ 3 & 2 & 0 \end{pmatrix}$. Find $[T]_{SS}$.

Exercise 2(15points, 5-5-5).
Let M and N two similar square matrices.
(1) Prove that M and N have the same determinant.
(2) Prove that for every real number λ, $det(M - \lambda I) = det(N - \lambda I)$ (det. is the determinant)
(3) Express N^n in terms of M^n.

Exercise 3(20points, 5-5-5-5).
Let $V = \mathbb{R}^4$ be the standard inner product space $u = (1, 1, 1, 1)$ and $v = (8, 2, 2, 0)$
(1) Determine the angle θ between u and v.
(2) Find the scalar projection and the vector projection z of u onto v.
(3) Verify that $u - z$ is orthogonal to z.
(4) Compute $\|u - z\|$, $\|z\|$ and $\|u\|$ and verify Pythagore's law .

Exercise 4 (15points, 5-5-5).
Let $V = \mathbb{R}^3$ be the standard inner product space and W the subspace of V spanned by $u = (1, 2, 1)$.
(1) Find the orthogonal subspace $W^\perp$ of W.
(2) Find a basis B of $W^\perp$ and its dimension.
(3) Prove that $V = W \bigoplus W^\perp$.

Exercise 5(10points, 5-5).
Let $V = \mathbb{R}^3$ endowed with a mapping defined by $(u|v) = x_1y_1 + 2x_2y_2 + 3x_3y_3$ for every $u = (x_1, x_2, x_3)$ and $v = (y_1, y_2, y_3)$.
(1) Verify that $(.|.)$ is an inner product on V.
(2) Let $S = \{e_1, e_2, e_3\}$ be the standard basis of V. Express $(u|v)$ in the form u^tAv for a matrix A to be determined.

Exercise 6(15points, 5-5-5).
Let $(V, (.|.)$ be an inner product space of finite dimension n and W a subspace of V.
(1) Prove that any finite family of orthogonal vectors is linearly independent.
(2) Prove that $W^{\perp}$ is a subspace of V.
(3) Prove that if B_1 is a basis of W and B_2 is a basis of $W^{\perp}$, then $B_1 \cup B_2$ is a basis of V.

Exercise 7(15points, 5-10).
Let $V = \mathbb{R}^3$ be the standard inner product space and $B_1 = \{(1,1,1),(1,0,1),(0,1,1)\}$.
(1) Verify that B_1 is a basis for V.
(2) Use Gram-Shmidt process to find an orthonormal basis B_2 of V.