Department of Mathematics and Statistics, KFUPM
Math 280, Term 172
Exam 2, March 18, 2018, Duration: 120 minutes

Name: ID:

Exercise 1(10points, 5-5).
Determine whether the given vectors span $\mathbb{R}^3$ or not. Justify.
(1) $v_1 = (1, 2, 1), v_2 = (1, 0, 1), v_3 = (2, 0, 1)$.
(2) $v_1 = (0, 2, 1), v_2 = (1, 1, 1), v_3 = (1, 5, 3), v_4 = (2, 4, 3)$.

Exercise 2(10points, 5-5).
Which one of the following subsets is a subspace of the corresponding vector space.
(1) $W = \{(a, b, c) \in \mathbb{R}^3 | a + b = 2c\}$.
(2) $W = \{(a, b, c) \in \mathbb{R}^3 | a^2 + b^2 + c^2 \leq 1\}$.

Exercise 3(20points, 5-5-5-5).
Let $V = \mathbb{R}$ be the vector space over the field $\mathbb{Q}$ of rational numbers and
$W =:= \{a + b\sqrt{2} + c\sqrt{3} | a, b, c \in \mathbb{Q}\}$.
(1) Prove that W is a subspace of V.
(2) Prove that $\{1, \sqrt{2}, \sqrt{3}\}$ are linearly independent.
(3) Find a basis and the dimension of W, $dim_{\mathbb{Q}}(W)$.
(4) Find a an infinite set of linearly independent vectors of V.

Exercise 4 (15points, 5-5-5).
Let $V = \mathbb{P}_2 := \{f \in \mathbb{R}[X] | deg(f) \leq 2\}$, $S = \{1, X, X^2\}$ the standard basis of V and $B = \{1 + X, 1 + X^2, 1 + X + X^2\}$.
(1) Prove that B is a basis for V.
(2) Find the transition matrix P from B to S.
(3) Find the coordinate vector of $f = 4 + 3X + 7X^2$ in the basis B.

Exercise 5(15points, 5-5-5).
Let V be a vector space over a field $\mathbb{F}$, B_1, B_2, B_3 three bases of V, P the transition matrix from B_2 to B_1 and Q the transition matrix from B_3 to B_2.
(1) Express the transition matrix N from B_3 to B_1 in terms of P and Q.
(2) Express the transition matrix M from B_1 to B_3 in terms of P and Q.
(3) Assume that $V = \mathbb{R}^3$, $P = \begin{pmatrix} 2 & 3 & 2 \\ 2 & -1 & 4 \\ 3 & 2 & 0 \end{pmatrix}$, $Q = \begin{pmatrix} 1 & 0 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ and $[x]_{B_3} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. Express $[x]_{B_1}$.

Exercise 6(15points, 5-5-5).
(1) Let A be an $n \times n$ matrix. Prove that A is invertible if and only if $rank(A) = n$.

Let $A = \begin{pmatrix} 2 & 3 & 2 & 4 \\ 2 & -1 & 4 & 4 \\ 3 & 2 & 0 & 6 \\ 1 & 0 & 1 & 2 \end{pmatrix}$.

(2) Find a basis and the dimension of the row space of A.
(3) Find a basis and the dimension of the row space of A.

Exercise 7(15points, 5-5-5-5).
Let $V = \mathbb{R}^3$ and $T : V \longrightarrow V$ defined by $T(a, b, c) = (b + c, a + c, a + b)$.
(1) Verify that T is a linear transformation.
(2) Find $ker(T)$, $dim(ker(T))$, $Im(T)$ and $dim(Im(T))$.
(3) Find the matrix representing T is the standard basis B of V.