Name: ID#: Serial #:

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1. [6pts] Let x_1, x_2, x_3 be integers such that $x_1 + x_2 + x_3 = 0$. Prove that at least one of x_1, x_2, x_3 is even.

Solution. Assume for contradiction that none of x_1, x_2, x_3 is even. Then $x_1 \equiv x_2 \equiv x_3 \equiv 1 (\bmod 2)$ and hence $0 \equiv x_1 + x_2 + x_3 \equiv 3 (\bmod 2)$, a contradiction. ■

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2. [12pts] (a) Prove that $\sqrt{3}$ is an irrational number.
(b) Prove that there is no rational number x such that $\frac{2}{x} - x = 2$.

Solution. (a) Suppose for contradiction that $\sqrt{3}$ is rational. Then $\sqrt{3} = \frac{a}{b}$ for some integers a, b where $\frac{a}{b}$ is in simplest form (i.e. that $\gcd(a, b) = 1$). We get $3b^2 = a^2$ so that $3|a^2$. Assume for contradiction that $3 \not| a$. Then $a \equiv \pm 1 \, (\bmod 3)$ and so $a^2 \equiv 1 \, (\bmod 3)$ contradicting that $3|a^2$. We therefore have $a = 3c$ for some integer c. Hence $3b^2 = 9c^2$, i.e. $3|b^2$ and we obtain (as above) that $3|b$. Hence 3 is a common divisor of a and b, contradicting that $\frac{a}{b}$ is in simplest form. ■
[*Another way*: Suppose for contradiction that $\sqrt{3}$ is rational. Then $\sqrt{3} = \frac{a}{b}$ for some integers a, b where $\frac{a}{b}$ is in simplest form (i.e. that $\gcd(a, b) = 1$). We get $3b^2 = a^2$. If b is even, then so too is a and this is impossible since $\frac{a}{b}$ is in simplest form. Hence b is odd and then so too is a. Put $a = 2u + 1, \ b = 2v + 1$ where u, v are integers. We then have $3(2v + 1)^2 = (2u + 1)^2$, i.e. $2(u^2 + u - 3v^2 - 3v) = 1$, which is impossible.■]

(b) Suppose for contradiction that $\frac{2}{x} - x = 2$ for some rational number x. Then $x^2 + 2x = 2$ i.e. $(x + 1)^2 = 3$. This means $\sqrt{3} = |x + 1|$ which is impossible since $\sqrt{3}$ is irrational by (a) and $|x + 1|$ is rational (as x is assumed to be rational). ■

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3. [8pts] The Fibonacci sequence $\{f_n\}$ is defined recursively by $f_1 = f_2 = 1, \quad f_n = f_{n-1} + f_{n-2}$ for each integer $n \geq 3$. Use induction to prove, for each positive integer n, that $f_1 + f_3 + \cdots + f_{2n-1} = f_{2n}$.

Solution. We have $f_1 = f_2$, hence the statement to prove is true for $n = 1$.
Suppose the statement to prove is true for $n = k$, where k is some positive integer. We have

$$f_1 + f_3 + \cdots + f_{2(k+1)-1} = f_1 + f_3 + \cdots + f_{2k-1} + f_{2k+1} = f_{2k} + f_{2k+1} = f_{2(k+1)}.$$

This proves that the statement is true for $n = k + 1$. ■

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4. [12pts] (a) A relation R is defined on $\mathbb{N}$ by aRb if $a|b^2$. Is R reflexive, symmetric, transitive? Justify.

(b) Compute $[100^{100}]$ in $\mathbb{Z}_7$.

Solution. (a) R is reflexive since $a|a^2$ for each a in $\mathbb{N}$. R is not symmetric because $1|2^2$ but $2 \not| 1^2$. R is not transitive because $8|4^2$ and $4|2^2$ but $8 \not| 2^2$.

(b) $100 \equiv 2 \pmod 7$ (since $100 = 14 \times 7 + 2$), hence $100^{100} \equiv 2^{100} \equiv (2^3)^{33} \times 2 \equiv 1^{33} \times 2 \equiv 2 \pmod 7$. So $[100^{100}] = [2]$ in $\mathbb{Z}_7$.

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5. [12pts] Consider the function $f : \mathbb{R} - \{-2/3\} \longrightarrow \mathbb{R} - \{-2/3\}$ given by $f(x) = \dfrac{2x+1}{-3x-2}$.

(a) Compute $f \circ f$.

(b) Prove that f is a bijection.

(c) Determine the set $f(\{0,2\})$ and then prove that if E is the set of all even integers, then $f(E)$ does not contain any integer.

Solution. (a) $(f \circ f)(x) = f(f(x)) = \dfrac{2\left(\dfrac{2x+1}{-3x-2}\right)+1}{-3\left(\dfrac{2x+1}{-3x-2}\right)-2} = \dfrac{2(2x+1)+(-3x-2)}{-3(2x+1)-2(-3x-2)} = x$. Hence $f \circ f$ is the identity map on $\mathbb{R} - \{-2/3\}$.

(b) From (a), f is its own inverse function. Since f has an inverse function, it must be injective.

(c) We have $f(0) = -\dfrac{1}{2}$ and $f(2) = -\dfrac{5}{8}$. Hence $f(\{0,2\}) = \left\{-\dfrac{1}{2}, -\dfrac{5}{8}\right\}$.

Let $y \in f(E)$. Then $y = f(x)$ for some even integer $x = 2t$ say, where $t \in \mathbb{Z}$. We have $y = \dfrac{2x+1}{-3x-2} = -\dfrac{4t+1}{2(3t+1)}$. Since $2 \nmid (4t+1)$, y cannot be an integer. ■

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6. [10pts] (a) Let $A = \{1,2\}$. Give an example of subsets A_1, A_2 of A and a function $f : A \longrightarrow A$ such that $f(A_1 \cap A_2) \neq f(A_1) \cap f(A_2)$.

(b) Let $g : A \longrightarrow B$ be a function and let A_1, A_2 be subsets of A. Prove that $g(A_1 \cap A_2) \subseteq g(A_1) \cap g(A_2)$.

Solution. (a) Let $A_1 = \{1\}$, $B = \{2\}$ and $f : A \longrightarrow A$ be the function given by $f(1) = f(2) = 1$. We have $A_1 \cap A_2 = \emptyset$ and $f(A_1) = f(A_2) = \{1\}$. Hence $f(A_1 \cap A_2) = \emptyset \neq \{1\} = f(A_1) \cap f(A_2)$.

(b) Let $y \in g(A_1 \cap A_2)$. Then $y = g(x)$ for some x in $A_1 \cap A_2$. Since $g(x)$ is in $g(A_1)$ and in $g(A_2)$, we obtain $y \in g(A_1) \cap g(A_2)$. This proves $g(A_1 \cap A_2) \subseteq g(A_1) \cap g(A_2)$. ■

[Note that the reverse inclusion in Part (b) is not always true (by Part (a)).]