Name: ID#: Serial #:

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1. [10pts] Let P, Q, R be statements. Is the logical equivalence

$$P \longrightarrow (Q \longrightarrow R) \equiv (P \wedge Q) \longrightarrow R$$

true? Justify.

Solution. We can use a truth table to show that the logical equivalence is true. A shorter way is as follows.

$$P \longrightarrow (Q \longrightarrow R) \equiv \sim P \vee (\sim Q \vee R) \equiv (\sim P \vee \sim Q) \vee R \equiv \sim (P \wedge Q) \vee R \equiv (P \wedge Q) \longrightarrow R$$

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2. [10pts] Let A, B, C be sets, $A \neq \emptyset$.

(a) Suppose $A \cap B = A \cap C$. Is it true that $B = C$? Justify.

Solution. No, take $A = B = \{0\}$, $C = \{0, 1\}$, then $A \cap B = A \cap C = \{0\}$ but $B \neq C$.

(b) Suppose $A \times B = A \times C$. Prove that $B = C$.

Proof. We first prove that $B \subseteq C$. Let $b \in B$. Since $A \neq \emptyset$, we can choose an element $a \in A$. We have $(a, b) \in A \times B$, so $(a, b) \in A \times C$, i.e. $b \in C$. Similarly, $C \subseteq B$. Hence $B = C$. ■

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3. [10pts] (a) Find subsets X, Y of $\{1, 2, 3\}$ such that $\mathcal{P}(X \cup Y) \nsubseteq \mathcal{P}(X) \cup \mathcal{P}(Y)$.

Solution. Take $X = \{1\}$, $Y = \{2\}$. Then $\{1, 2\} \in \mathcal{P}(X \cup Y)$ but $\{1, 2\} \notin \mathcal{P}(X)$ (since $2 \notin X$) and $\{1, 2\} \notin \mathcal{P}(Y)$ (since $1 \notin Y$).

(b) Let A, B be sets. Prove that $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$.

Proof. Let $S \in \mathcal{P}(A) \cup \mathcal{P}(B)$. Suppose first that $S \in \mathcal{P}(A)$, then $S \subseteq A \subseteq A \cup B$, so that $S \in \mathcal{P}(A \cup B)$.
Similarly, if $S \in \mathcal{P}(B)$, we obtain $S \in \mathcal{P}(A \cup B)$. This proves $\mathcal{P}(A)$ and $\mathcal{P}(B)$ are both subsets of $\mathcal{P}(A \cup B)$ so that $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$. ■

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4. [10pts] Let x, y be real numbers.

(a) Using properties of absolute value, prove that $|x + y| - |x - y| \leq 2|x|$.

Proof. $2|x| + |x - y| = |2x| + |y - x| \geq |2x + (y - x)| = |x + y|$. ■

(b) Prove that if $|x| < |y|$, then $x^2 - 2xy + 3y^2 \geq 0$. Is it true that $x^2 - 2xy + 3y^2 > 0$? Justify.

Solution.

- *Proof.* We have $x^2 - 2xy + 3y^2 = (x - y)^2 + 2y^2 \geq 0$ for all real numbers. Hence the statement to prove is trivially true. ■
- It is true that $x^2 - 2xy + 3y^2 > 0$ when $|x| < |y|$: This is because $|x| < |y|$ implies $x \neq y$ i.e. $(x - y)^2 > 0$, hence $(x - y)^2 + 2y^2 > 0$ i.e. $x^2 - 2xy + 3y^2 > 0$.

  [**Note** that $|x| < |y|$ also implies that $y^2 > 0$ and also that if we do not assume $|x| < |y|$, then $x^2 - 2xy + 3y^2 > 0$ is not always true: take $x = y = 0$.]

5. [10pts] Let x, y be integers.

(a) Prove that if $4|\left(x^2+y^2\right)$, then x and y are even.

Proof. We use the contrapositive and prove that if x and y have different parity or if both are odd, then x^2+y^2 is not divisible by 4.
If x and y have different parity, then x^2 and y^2 also have different parity and so x^2+y^2 is odd and cannot be divisible by 4.
If x and y are odd, then there are integers h, k such that $x=2h+1$, $y=2k+1$. In this case, $x^2+y^2=4\left(k^2+k+h^2+h\right)=2$, which is not divisible by 4. ■

(b) Prove that if $3|5x$, then $3|x$.

Proof. If $3|5x$, then $3|(6x-5x)$ (because $3|6x$), i.e. $3|x$. ■

[**Note**. This problem can also be solved using congruences mod 4 in (a) and mod 3 in (b).]

6. [10pts] Let $a, b \in \mathbb{Z}$.

(a) Prove that if $a \equiv 2 \pmod 6$ and $b \equiv 1 \pmod 4$, then $2a-3b \equiv 1 \pmod{12}$.

Proof. If $a \equiv 2 \pmod 6$ and $b \equiv 1 \pmod 4$, then $2a \equiv 4 \pmod{12}$ and $3b \equiv 3 \pmod{12}$. Hence $2a-3b \equiv 1 \pmod{12}$. ■

(b) Prove that $5 \nmid a$ if and only if $a^4 \equiv 1 \pmod 5$.

Proof. Suppose $5 \nmid a$. Then $a \equiv \pm 1 \pmod 5$ or $a \equiv \pm 2 \pmod 5$. In the first case, $a^4 \equiv 1 \pmod 5$; and in the second case, $a^4 \equiv 16 \equiv 1 \pmod 5$.
Conversely, if $a^4 \equiv 1 \pmod 5$, then clearly a is not divisible by 5 (otherwise, $a \equiv 0 \pmod 5$ and then $a^4 \equiv 0 \pmod 5$). ■