KFUPM – Department of Mathematics & Statistics

MATH202 – Term 172

Quiz 6A

Name: Solution ID #: Section: 16

Q1 (**3.5 Points**) Find the general solution of the system

$$\mathbf{X}' = \begin{pmatrix} 3 & -4 & 0 \\ 1 & 1 & 2 \\ 0 & 2 & 3 \end{pmatrix} \mathbf{X}$$

Hint: *eigenvalues of the coefficients matrix are* $\lambda_1 = \lambda_2 = 3,\ \lambda_3 = 1.$

Solution: For $\lambda_3 = 1$

$$\left(\begin{array}{ccc|c} 2 & -4 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 2 & 0 \end{array}\right) \xrightarrow{\text{Row operations}} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

So, $k_1 = -2k_3$ and $k_2 = -k_3$.

Putting $k_3 = -1$ gives

$$K_1 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

For $\lambda_2 = \lambda_1 = 3$, we have

$$\left(\begin{array}{ccc|c} 0 & -4 & 0 & 0 \\ 1 & -2 & -2 & 0 \\ 0 & 2 & 0 & 0 \end{array}\right) \xrightarrow{\text{Row operations}} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

So, $k_2 = 0$, $k_1 = -2k_3$. Putting $k_3 = 1$ gives

$$K_2 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

To find P, we solve $(A - \lambda_1 I)P = K_2$:

$$\left(\begin{array}{ccc|c} 0 & -4 & 0 & -2 \\ 1 & -2 & 2 & 0 \\ 0 & 2 & 0 & 1 \end{array}\right) \xrightarrow{\text{Row operations}} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

we have, $P_2 = 1/2$ and $P_1 + 2P_3 = 1$.

choosing $P_3 = 0$ gives $P_1 = 1$. Thus,

$$P = \begin{pmatrix} 1 \\ 1/2 \\ 0 \end{pmatrix}$$

The general solution is

$$X = c_1 \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} e^{3t} + c_3 \left[\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} te^{3t} + \begin{pmatrix} 1 \\ 1/2 \\ 0 \end{pmatrix} e^{3t} \right]$$

Q2 (**3 Points**) Use variation of parameters to solve the system

$$\frac{dx}{dt} = y + 1$$

$$\frac{dy}{dt} = -x + \cot t$$

Solution: The system can be written as

$$X' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} X + \begin{pmatrix} 1 \\ \cot t \end{pmatrix}.$$

The characteristic equation is $\begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0.$

Eigenvalues are $\lambda_1 = i$, $\lambda_2 = -i$.

For $\lambda_1 = i$, we have $\begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$

Thus, $k_2 = i k_1 \implies K_1 = \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Put $k_1 = 1$

$B_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $B_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$

$$X_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\cos t - \begin{pmatrix} 0 \\ 1 \end{pmatrix}\sin t = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}$$

$$X_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\cos t + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\sin t = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$

The fundamental matrix $\phi(t) = (X_1\ X_2) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$

$\phi^{-1}(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$ and $\phi^{-1}(t)F(t) = \begin{pmatrix} 0 \\ \sin t + \dfrac{\cos^2 t}{\sin t} \end{pmatrix}$

$$X_p = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \int \begin{pmatrix} 0 \\ \cancel{\sin t} + \csc t - \cancel{\sin t} \end{pmatrix} dt$$

$$= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} 0 \\ -\ln|\csc t + \cot t| \end{pmatrix}$$

$$= \begin{pmatrix} -\sin t \ln|\csc t + \cot t| \\ -\cos t \ln|\csc t + \cot t| \end{pmatrix}$$

The general solution is

$$X = c_1 X_1 + c_2 X_2 + X_p$$

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Quiz **6**A

Q3 (**3.5 Points**) Let

$$\mathbf{A} = \begin{pmatrix} -1 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & 1 & 1 \end{pmatrix}.$$

(a) Find $e^{\mathbf{A}t}$

(b) Use part (a) to find general solution of the system $\mathbf{X}' = \mathbf{AX}$

Solution:

$$A^2 = \begin{pmatrix} -1 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & 1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & +1 & 1 \\ -1 & 0 & 1 \\ -1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(a)
$$e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \dots$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} -t & t & t \\ -t & 0 & t \\ -t & t & t \end{pmatrix} + \begin{pmatrix} -\frac{t^2}{2} & 0 & \frac{t^2}{2} \\ 0 & 0 & 0 \\ -\frac{t^2}{2} & 0 & \frac{t^2}{2} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1-t-\frac{t^2}{2} & t & t+\frac{t^2}{2} \\ -t & 1 & t \\ -t-\frac{t^2}{2} & t & 1+t+\frac{t^2}{2} \end{pmatrix}$$

(b) The general solution is

$$X = e^{At}C = e^{At}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

$$= \begin{pmatrix} c_1\left(1-t-\frac{t^2}{2}\right) + c_2t + c_3\left(t+\frac{t^2}{2}\right) \\ -c_1t + c_2 + c_3t \\ c_1\left(-t-\frac{t^2}{2}\right) + c_2t + c_3\left(1+t+\frac{t^2}{2}\right) \end{pmatrix}$$