KFUPM – Department of Mathematics & Statistics

MATH202 – Term 172

Quiz 5A (Duration: 25 Minutes)

Name: Solution ID #: Section: 16

Q1 (2 Points) Determine the singular points of the DE

$$y'' - \frac{y'}{(x-1)} + \frac{y}{(x-2)^3} = 0.$$

Classify each singular point as regular or irregular.

$P(x) = -\frac{1}{x-1}$ is not analytic at $x=1$.

$Q(x) = \frac{1}{(x-2)^3}$ is not analytic at $x=2$.

So, the singular points are $x=1$ and $x=2$.

For $x=1$,

$p(x) = (x-1)P(x) = -1$

$q(x) = (x-1)^2 Q(x) = \frac{(x-1)^2}{(x-2)^3}$

are both analytic at $x=1$.

So, $x=1$ is a regular singular point.

For $x=2$, $q(x) = (x-2)^2 Q(x) = \frac{1}{x-2}$

is not analytic at $x=2$

So, $x=2$ is a irregular singular point.

Q2 (3.5 Points) $x=0$ is a regular singular point of the DE $2xy'' - (3+2x)y' + y = 0$. Use the general form of the indicial equation to find the indicial roots of the singularity. Without solving, discuss the number of series solutions you would expect to find using the method of Frobenius.

$P(x) = -\frac{3+2x}{2x} = -\frac{3}{2x} - 1$

$Q(x) = \frac{1}{2x}$

at $x=0$,

$p(x) = x\,P(x) = -\frac{3}{2} - x$

$q(x) = x^2 Q(x) = \frac{x}{2}$

$a_0 = -\frac{3}{2}, \quad b_0 = 0$

The indicial equation:

$r(r-1) - \frac{3}{2}r = 0$

$r\left(r - \frac{5}{2}\right) = 0$

$r_1 = \frac{5}{2}, \quad r_2 = 0,$

$r_1 - r_2 = \frac{5}{2}$ is not a positive integer.

Thus, we expect to find two linearly independent series solution using the method of Frobenius.

Q3 (4.5 Points) Find the first three nonzero terms in each of two linearly independent series solutions of the DE $y'' + 3xy' + 3y = 0$ about the ordinary point $x = 0$.

$$y = \sum_{n=0}^{\infty} c_n x^n$$

$$y' = \sum_{n=1}^{\infty} c_n n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2}$$

$$y'' + 3xy' + 3y = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2} + 3x \sum_{n=1}^{\infty} c_n n x^{n-1} + 3 \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2} + \sum_{n=1}^{\infty} 3 c_n n x^n + \sum_{n=0}^{\infty} 3 c_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2} + \sum_{n=1}^{\infty} (3 c_n n + 3 c_n) x^n + 3c_0 = 0$$

$$\Rightarrow 3c_0 + 2c_2 + \underbrace{\sum_{n=3}^{\infty} n(n-1) c_n x^{n-2}}_{\text{Put } k=n-2} + \underbrace{\sum_{n=1}^{\infty} 3(n+1) c_n x^n}_{\text{Put } k=n} = 0$$

$$\Rightarrow 3c_0 + 2c_2 + \sum_{k=1}^{\infty} (k+2)(k+1) c_{k+2} x^k + \sum_{k=1}^{\infty} 3(k+1) c_k x^k = 0$$

$$\Rightarrow 3c_0 + 2c_2 + \sum_{k=1}^{\infty} \left[(k+2)(k+1) c_{k+2} + 3(k+1) c_k\right] x^k = 0$$

$$c_2 = -\frac{3}{2} c_0.$$

$$c_{k+2} = -\frac{3c_k}{k+2}$$

$k=1 \Rightarrow c_3 = -c_1$

$k=2 \Rightarrow c_4 = -\frac{3}{4} c_2 = \frac{9}{8} c_0$

$k=3 \Rightarrow c_5 = -\frac{3c_k}{5} = \frac{3}{5} c_1$

$$y_1 = 1 - \frac{3}{2}x^2 + \frac{9}{8}x^4 - \dots$$

$$y_2 = x - x^3 + \frac{3}{5}x^5 - \dots$$