Math 202 – Quiz 6 (Term 172)

Name: **ID #:**

Problem 1 (5 points):

Solve the following system of differential equations

$$X'(t) = \begin{pmatrix} -1 & 3 \\ -3 & 5 \end{pmatrix} X(t)$$

<u>Sol.</u> Eigenvalues

$$\begin{vmatrix} -1-\lambda & 3 \\ -3 & 5-\lambda \end{vmatrix} = \lambda^2 - 4\lambda - 5 + 9 = (\lambda - 2)^2 = 0$$

$\Rightarrow \lambda = 2, 2.$

<u>Eigenvectors</u> $\begin{pmatrix} -3 & 3 \\ -3 & 3 \end{pmatrix} \xrightarrow[R_1/-3]{R_2 - R_1} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}.$

$\Rightarrow \quad x_1 - x_2 = 0 \Rightarrow \quad E_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector

$\Rightarrow X_1 = \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix}$ is a solution.

<u>Second solution</u> $\quad X_2(t) = E_1 t e^{2t} + E_2 e^{2t}$

Such that $(A - 2I)K_2 = K_1$

$$\left(\begin{array}{cc|c} -3 & 3 & 1 \\ -3 & 3 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & -1 & -\frac{1}{3} \\ 0 & 0 & 0 \end{array}\right) \Rightarrow \quad x_1 - x_2 = -\frac{1}{3}$$

$\Rightarrow x_2 = x_1 + \frac{1}{3} \quad \Rightarrow E_2 = \begin{pmatrix} 0 \\ 1/3 \end{pmatrix}.$

$$\Rightarrow X_2 = \begin{pmatrix} te^{2t} \\ te^{2t} \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{3} e^{2t} \end{pmatrix} = \begin{pmatrix} te^{2t} \\ te^{2t} + \frac{1}{3} e^{2t} \end{pmatrix}$$

The solution is

$$X = c_1 X_1 + c_2 X_2$$

Problem 2 (5 points): Solve the following system of differential equations

$$\begin{cases} x'(t) &= 5x(t) + y(t) \\ y'(t) &= -2x(t) + 3y(t) \end{cases}$$

Sol. $X'(t) = \begin{pmatrix} 5 & 1 \\ -2 & 3 \end{pmatrix} X(t), \quad X(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$

Eigenvalues $\begin{vmatrix} 5-\lambda & 1 \\ -2 & 3-\lambda \end{vmatrix} = \lambda^2 - 8\lambda + 17 = 0$

$\Rightarrow \lambda = \dfrac{8 \pm 2i}{2} = 4 \pm i$ (complex).

Eigenvectors $\boxed{\lambda = 4+i}$

$\begin{pmatrix} 1-i & 1 \\ -2 & -1-i \end{pmatrix} \xrightarrow[R_2 + \frac{2R_1}{1-i}]{R_1} \begin{pmatrix} 1-i & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow (1-i)x_1 + x_2 = 0$

$\Rightarrow x_2 = \dfrac{-1}{1-i} x_1 = \dfrac{-(1+i)}{2} x_1 \Rightarrow K_1 = \begin{pmatrix} 2 \\ -1-i \end{pmatrix}$

is an eigenvector.

$\boxed{\lambda = 4-i} \Longrightarrow K_2 = \overline{K_1} = \begin{pmatrix} 2 \\ -1+i \end{pmatrix}$ is an eigen vector.

$K_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix} + i \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

Thus $X_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{4t} \cos t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{4t} \sin t$

$X_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{4t} \sin t + \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{4t} \cos t$

The general solution is

$$X = c_1 X_1 + c_2 X_2.$$