Math 202 – Quiz 4 (Term 172)

Name: **ID #:**

Problem 1 (3.5 points):

If $y_1 = \cos(lnx)$ and $y_2 = \sin(lnx)$ form a fundamental set of solutions to the differential equation $x^2y'' + xy' + y = 0$, find the general solution of

$$x^2y'' + xy' + y = \sec(lnx).$$

Sol. standard form $y'' + \frac{y'}{x} + y = \frac{\sec(\ln x)}{x^2}$,

The general solution is $y = c_1 y_1 + c_2 y_2 + y_p$.

$y_p = u_1(x) y_1 + u_2(x) y_2$ such that

$$u_1' = \frac{\begin{vmatrix} 0 & \sin(\ln x) \\ \frac{\sec(\ln x)}{x^2} & \frac{\cos(\ln x)}{x} \end{vmatrix}}{\begin{vmatrix} \cos(\ln x) & \sin(\ln x) \\ -\frac{\sin(\ln x)}{x} & \frac{\cos(\ln x)}{x} \end{vmatrix}} = \frac{-\sin(\ln x)\sec(\ln x)/x^2}{\frac{1}{x}}$$

$$= \frac{-\sin(\ln x)\sec(\ln x)}{x} = -\frac{\sin(\ln x)}{x\cos(\ln x)}$$

$\Rightarrow u_1 = \ln|\cos(\ln x)|$

$$u_2' = \frac{\begin{vmatrix} \cos(\ln x) & 0 \\ -\frac{\sin(\ln x)}{x} & \frac{\sec(\ln x)}{x^2} \end{vmatrix}}{\frac{1}{x}} = \frac{\cos(\ln x)\sec(\ln x)}{x}$$

$$= \frac{1}{x} \implies u_2 = \ln x \,, \quad x > 0 .$$

Thus $y_p = \cos(\ln x)\ln|\cos(\ln x)| + \sin(\ln x)\ln x$

The general solution is

$$y = y_c + y_p$$
$$= c_1 \cos(\ln x) + c_2 \sin(\ln x) + y_p .$$

Problem 2 (3.5 points): Use the Annihilator method to find a particular solution of the differential equation

$$y'' - y' - 2y = cosxsinx.$$

Homogeneous

$$y'' - y' - 2y = 0$$

$$m^2 - m - 2 = (m-2)(m+1) = 0$$

$$\Rightarrow \quad y_c = c_1 e^{-x} + c_2 e^{2x}.$$

Nonhomogeneous

$$(D^2 - D - 2)y = \frac{1}{2}\sin 2x$$

$D^2 + 4$ annihilates $\sin 2x$ and $\cos 2x$

$$\Rightarrow \quad (D^2+4)(D-2)(D+1)y = (D^2+4)\left(\frac{1}{2}\sin 2x\right) = 0$$

$$\Rightarrow \quad y_H = c_1 e^{-x} + c_2 e^{2x} + c_3 \sin 2x + c_4 \cos 2x.$$

Thus $\quad y_p = a \sin 2x + b \cos 2x, \quad a, b$ to be determined

$$y_p' = 2a \cos 2x - 2b \sin 2x$$

$$y_p'' = -4a \sin 2x - 4b \cos 2x$$

Substitute in the equation

$$-4a\sin 2x - 4b\cos 2x - 2a\cos 2x + 2b\sin 2x$$
$$- 2a\sin 2x - 2b\cos 2x = \frac{1}{2}\sin 2x$$

$$(2b-6a)\sin 2x - (6b+2a)\cos 2x = \frac{1}{2}\sin 2x$$

$\Rightarrow \quad 2a+6b=0 \Leftrightarrow a=-3b$

$2b-6a=\frac{1}{2} \Leftrightarrow 2b+18b = 20b = \frac{1}{2} \Rightarrow \boxed{b=\frac{1}{40}}$

$\Rightarrow \boxed{a = -\frac{3}{40}}$

So $y_p = -\frac{3}{40}\sin 2x + \frac{1}{40}\cos 2x$.

The solution of the nonhomogeneous is

$$y = y_c + y_p.$$

3.5

Problem 3 (● points): Find the general solution of the differential equation

$$y''' + 3y'' + 2y' = \frac{1}{x}, \qquad x > 0.$$

Sol. Homogeneous $y''' + 3y'' + 2y' = 0$

Auxiliary $m^3 + 3m^2 + 2m = m(m+1)(m+2) = 0$

$\Rightarrow m = -2, -1, 0.$

So $y_c = c_1 + c_2 e^{-x} + c_3 e^{-2x}.$

Nonhomog. A particular solution is

$y_p = u_1 + u_2 e^{-x} + u_3 e^{-2x}$, where

$$u_1' = \frac{\begin{vmatrix} 0 & e^{-x} & e^{-2x} \\ 0 & -e^{-x} & -2e^{-2x} \\ \frac{1}{x} & e^{-x} & 4e^{-2x} \end{vmatrix}}{\begin{vmatrix} 1 & e^{-x} & e^{-2x} \\ 0 & -e^{-x} & -2e^{-2x} \\ 0 & e^{-x} & 4e^{-2x} \end{vmatrix}} = \frac{\frac{1}{x}(-e^{-3x})}{-6e^{-3x}} = \frac{1}{6x}$$

$\Rightarrow u_1 = \frac{1}{6} \ln x.$

$$u_2' = \frac{\begin{vmatrix} 1 & 0 & e^{-2x} \\ 0 & 0 & -2e^{-2x} \\ 0 & \frac{1}{x} & 4e^{-2x} \end{vmatrix}}{-6e^{-3x}} = \frac{2e^{-2x}}{-6xe^{-3x}} = -\frac{e^{x}}{3x}$$

$\Rightarrow u_2 = -\frac{1}{3}\int \frac{e^x \, dx}{x}.$

$$u_3' = \frac{\begin{vmatrix} 1 & e^{-x} & 0 \\ 0 & -e^{-x} & 0 \\ 0 & e^{-x} & \frac{1}{x} \end{vmatrix}}{-6e^{-3x}} = \frac{e^{2x}}{6x}$$

$$\Rightarrow \quad u_3 = \int \frac{e^{2x}\,dx}{6x}$$

So

$$y_p = \frac{1}{6}\ln x - \frac{e^{-x}}{3}\int \frac{e^{x}\,dx}{x} + \frac{e^{-2x}}{6}\int \frac{e^{2x}\,dx}{x}.$$

The solution of the nonhomog. is

$$y = y_c + y_p.$$