Name:

KFUPM ID:

Exercise 1

Consider the following differential equation

$$y' - y^2 + 3y - 2 = 0. \tag{1}$$

1. Show that
$$y(x) = \frac{2 - c\,e^x}{1 - c\,e^x},$$
is a one-parameter family of solutions of the differential equation (1).
2. Find a singular solution of the differential equation (1).

Solution

1. We have to check that $y(x)$ satisfies the Differential equation (1). We write

$$y'(x) = \frac{-ce^x(1-ce^x) + ce^x(2-ce^x)}{(1-ce^x)^2} = \frac{ce^x}{(1-ce^x)^2}.$$

Now,

$$\begin{aligned} y' - y^2 + 3y &= \frac{ce^x - (2-ce^x)^2 + 3(2-ce^x)(1-ce^x)}{(1-ce^x)^2} \\ &= \frac{ce^x - 4 - c^2e^{2x} + 4ce^x + 6 + 3c^2e^{2x} - 9ce^x)}{(1-ce^x)^2} \\ &= \frac{2c^2e^{2x} - 4ce^x + 2}{(1-ce^x)^2} = \frac{2\,(1-ce^x)^2}{(1-ce^x)^2} = 2. \end{aligned}$$

That is

$$y' - y^2 + 3y - 2 = 0.$$

In summary, $y(x) = \frac{2-c\,e^x}{1-c\,e^x}$ is a one-parameter family of solutions of the differential equation (1).

2. Let's look for constant solutions of the differential equation (1). Let $y = k$ a solution of (1), then $-k^2 + 3k - 2 = 0$. It is rather easy to see that $k = 1$ or $k = 2$. Thus $y = 1$ and $y = 2$ are solutions of the differential equation (1). Now, observe that if we set $c = 0$ in the expression

$$y(x) = \frac{2 - c\,e^x}{1 - c\,e^x},$$

we obtain $y = 2$. Thus $y = 2$ is not a singular solution. Now, let us find c so that

$$1 = \frac{2 - c\,e^x}{1 - c\,e^x} \iff 2 - c\,e^x = 1 - c\,e^x \iff 2 = 1 \quad \text{Absurd 1.}$$

Eventually, we deduce that $y = 1$ is a singular solution of the differential equation (1).

Exercise 2

Consider the following initial value problem

$$\begin{cases} y' = \sqrt{y^2-4} + \sqrt{9-x^2}, \\ \\ y(x_0) = y_0. \end{cases} \tag{2}$$

1. Find and sketch the region of all $(x_0, y_0) \in \mathbb{R}^2$ such that the initial value problem (2) has a unique solution following the Theorem of existence uniqueness of solutions.
2. Find the largest interval on which the solution of the IVP (2) with $x_0 = 1$ and $y_0 = 3$ may be defined.

Solution

1. If we let

$$f(x,y) = \sqrt{y^2-4} + \sqrt{9-x^2},$$

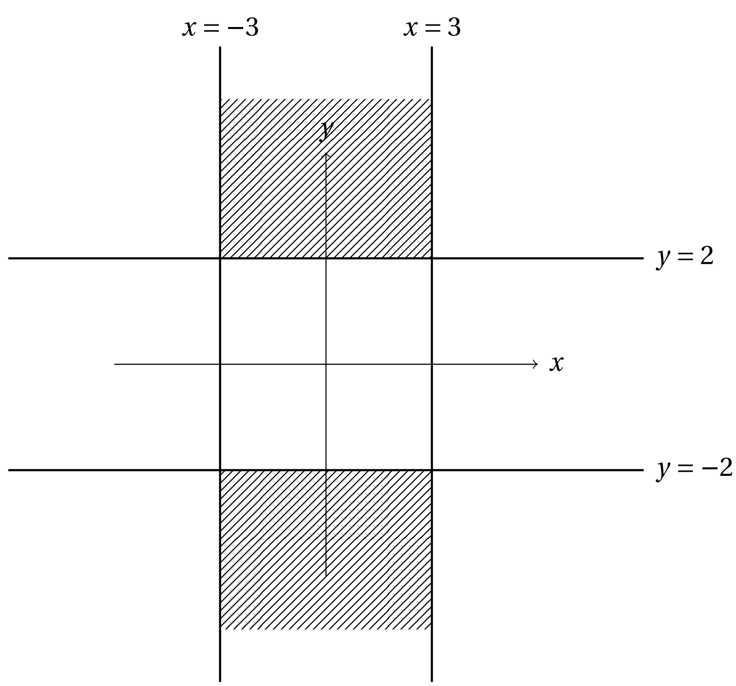
then, formally, we have

$$\frac{\partial f}{\partial y} = \frac{y}{\sqrt{y^2-4}}.$$

We see clearly that the region where f and $\frac{\partial f}{\partial y}$ are continuous is given by

$$\mathscr{R} = \left\{(x,y) \in \mathbb{R}^2 \mid \left\{(y < -2) \text{ or } (y > 2)\right\} \quad \text{and} \quad \{-3 \le x \le 3\}\right\}$$

Using the existence and uniqueness Theorem, the IVP (2) has a unique solution if $(x_0, y_0) \in \mathscr{R}$. The region $\mathscr{R}$ is the following



2. If $(x_0, y_0) = (1,3)$, then (x_0, y_0) is in the region $\mathscr{R}$. Thus the IVP (2) has a unique solution in an appropriate interval containing 1. The largest interval containing 1 on which the solution may be defined is

$$I = (-3, 3).$$

Exercise 3

Solve the following initial value problem

$$\begin{cases} y' - x\,e^{x+y} = 0, \\ y(0) = 0, \end{cases}$$

and find the largest interval on which the solution may be defined.

Solution

Separating variables, the differential equation may be written

$$e^{-y}\,dy = xe^{x}\,dx.$$

This differential equation is clearly separable. Integrating the latter equation, we get

$$\int e^{-y}\,dy = \int xe^{x}\,dx = xe^{x} - \int e^{x}\,dx = (x-1)\,e^{x} + c, \qquad \text{where} \quad c \in \mathbb{R}.$$

Thus, we have

$$-e^{-y} = (x-1)\,e^{x} + c \quad \Longleftrightarrow \quad e^{-y} = (1-x)\,e^{x} \quad \Longleftrightarrow \quad -y = \ln((1-x)\,e^{x} + c)$$

The initial condition $y(0) = 0$ leads to

$$0 = \ln((1-0)\,e^{0} + c) = \ln(1+c) \quad \Longleftrightarrow \quad c = 0.$$

Consequently, the solution of the IVP (2) is given by

$$y = -\ln((1-x)\,e^{x}) = \ln\frac{e^{-x}}{1-x}.$$

The largest interval on which the solution may be defined is

$$I = (-\infty, 1).$$

Exercise 4

1. Evaluate the derivative of the function $1+(\ln x)^2$.
2. Solve the following differential equation

$$x\left(1+(\ln x)^2\right) y' + 2\ln(x)\, y = 1, \tag{3}$$

on $(0,+\infty)$.

Solution

1. Clearly

$$\frac{d}{dx}(1+(\ln x)^2) = \frac{2\ln x}{x}.$$

2. The differential equation (3) is clearly first order linear differential equation. Its normal form is

$$y' + \frac{2\ln x}{x\left(1+(\ln x)^2\right)}\, y = \frac{1}{x\left(1+(\ln x)^2\right)}. \tag{4}$$

An integrating factor of (4) is given by

$$u(x) = e^{\int \frac{2\ln x}{x\left(1+(\ln x)^2\right)}\, dx}$$

Now, thanks to the first question of the exercise, we have

$$\int \frac{2\ln x}{x\left(1+(\ln x)^2\right)}\, dx = \int \frac{\frac{2\ln x}{x}}{\left(1+(\ln x)^2\right)}\, dx = \int \frac{(1+(\ln x)^2)'}{1+(\ln x)^2}\, dx = \ln(1+(\ln x)^2) + c, \quad c\in\mathbb{R}.$$

Since any c will do, we fix $c=0$. Therefore

$$u(x) = e^{\ln(1+(\ln x)^2)} = 1+(\ln x)^2,$$

is an integrating factor of the linear differential equation (4). Multiplying the differential equation (4) by $u(x)$, we get

$$(u(x)\,y(x))' = \frac{1+(\ln x)^2}{x\left(1+(\ln x)^2\right)} = \frac{1}{x}.$$

Integrating this equation, we obtain

$$(1+(\ln x)^2)\, y(x) = \ln|x| + c, \quad \text{where} \quad c\in\mathbb{R}.$$

Since we are solving the differential equation on $(0,+\infty)$, we end up with

$$y(x) = \frac{\ln|x|+c}{1+(\ln x)^2}, \quad \text{where} \quad c\in\mathbb{R}.$$

Eventually, $y(x)$ is a solution of the differential equation (3) on $(0,+\infty)$.