Additional Exercises

Math 202: Sections 1.1 & 1.2

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February 18, 2018

This is a list of exercises with corrections. We do not pretend that this list covers entirely the whole material of the indicated sections, and we warmly recommend to not stop practicing after finishing these exercises. Never forget that knowledge is a treasure, but practice is the key to it.

1 Statement

Exercise I

Show that

$$t = \ln\left(\frac{2x-1}{x-1}\right)$$

is an implicit solution of the DE

$$\frac{dx}{dt} = (x-1)(1-2x).$$

Give the explicit solution and its interval of definition.

Exercise II

Find and sketch teh region where the initial value problem

$$y' = x\sqrt{y}, \quad y(x_0) = y_0,$$

has a unique solution (following the existence and uniqueness Theorem).

Exercise III

Consider the DE

$$y' + 2xy^2 = 0, \tag{1}$$

- State the order of the DE (1).
- Classify the DE (1) in terms of linearity.
- Verify that

  $$y = \frac{1}{x^2+c},$$

  is a one-parameter family of the DE (1).
- Find the solution of the IVP

  $$y' + 2xy^2 = 0, \quad y(0) = -2. \tag{2}$$

- Give the largest interval on which the previous solution is defined.

Exercise IV

Verify that

$$y = \frac{1+c\,e^t}{1-c\,e^t}, \tag{3}$$

is a one-parameter family of solutions of the DE

$$y' = \frac{1}{2}(y^2-1).$$

Find a singular solution of the DE (3).

Exercise V

- Verify that

  $$y = x + \frac{c}{x},$$

  is a one-parameter family of solutions of the DE

  $$xy' + y = 2x.$$

- Deduce a solution of the IVP

  $$xy' + y = 2x, \quad y(-2) = 1, \tag{4}$$

  and find its interval of definition.

Exercise VI

Given that

$$y = \frac{c\,e^{-x}}{1-c\,e^{-x}},$$

is a one-paramter family of solutions of the DE

$$y'' + y^2 + y = 0, \tag{5}$$

find all singular constant solutions of the DE (5).

Exercise VII

Find and sketch the region where the IVP

$$\frac{dy}{dx} = \frac{\sqrt{y^2-4}}{\sqrt{x^2-4}}, \quad y(x_0) = y_0, \tag{6}$$

has a unique solution for all (x_0, y_0) following the existence and uniqueness Theorem.

Exercise VIII

Verify that

$$x = 1 + \frac{(3t^2+c)^3}{27}, \quad c \in \mathbb{R}, \tag{7}$$

is a one-parameter family of solutions of the DE

$$\frac{dx}{dt} = 6t(x-1)^{\frac{2}{3}}. \tag{8}$$

Find a singular constant solution of the DE (8).

Exercise IX

Given that $2x^2 - y^2 - 2x = c$ is a one-parameter family of solutions of the DE

$$y\,\frac{dy}{dx} = 2x - 1, \tag{9}$$

find an explicit solution which satisfies $y(1) = -2$.

Exercise X

Determine wether the existence and uniqueness Theorem guarantees that the IVP

$$\frac{dy}{dx} = xy^{\frac{2}{3}}, \quad y(1) = 0, \tag{10}$$

posses a unique solution or not.

Exercise XI

Verify that $x^2y^4 + x^3 - 27 = 0$ defines an implicit solution of the DE

$$4xy^3\,\frac{dy}{dx} + 2y^4 + 3x = 0, \tag{11}$$

on the interval $(0, 3)$.

Exercise XII

Find the values of b for which the IVP

$$\frac{dy}{dx} = \frac{\sqrt{y-6x}}{x^2+1}, \quad y(5) = b, \tag{12}$$

has a unique solution using the existence and uniqueness Theorem.

2 Correction

Exercise I

By implicit differentiation, we have

$$\begin{aligned} 1 &= \frac{d}{dt}\ln\left(\frac{2x-1}{x-1}\right) \\ &= \frac{d}{dt}\left(\ln(2x-1) - \ln(x-1)\right) \\ &= \left(\frac{2}{2x-1} - \frac{1}{x-1}\right)\frac{dx}{dt} \\ &= -\frac{1}{(2x-1)(x-1)}\,\frac{dx}{dt}. \end{aligned}$$

In particular,

$$\frac{dx}{dt} = (x-1)(1-2x).$$

To obtain an explicit solution $x(t)$, it is enough to inverse the expression

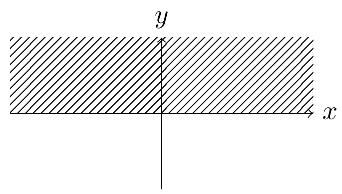
$$t = \ln\left(\frac{2x-1}{x-1}\right).$$

That is

$$\begin{aligned} e^t = \frac{2x-1}{x-1} &\iff x(e^t-2) = e^t - 1 \\ &\iff x = \frac{e^t-1}{e^t-2}. \end{aligned}$$

Exercise II

The conditions of the existence and uniqueness Theorem are: f and $\frac{\partial f}{\partial y}$ are continuous on an open rectangle containing (x_0, y_0) where $f(x,y) = x\sqrt{y}$. This conditions clearly translates to $y > 0$. Then, the required region is $\mathcal{R} = \{(x,y) \in \mathbb{R}^2 \,|\, y > 0\}$. That is the upper half plane not containing the $x-$axis.



Exercise III

- The DE is of first order.
- The DE involves a quadratic term in y, namely y^2, so it is clearly nonlinear.
- Let $y = \frac{1}{x^2+c}$, then for all $c \in \mathbb{R}$, we have clearly

  $$y' + 2xy^2 = \frac{-2x}{(x^2+c)^2} + \frac{2x}{(x^2+c)^2} = 0.$$

  In particular, $y = \frac{1}{x^2+c}$ is a one-parameter family of solutions to the DE (1).

- We have
$$y(0) = -2 \iff \frac{1}{c} = -2 \iff c = -2.$$
That is $y = \frac{1}{x^2-2}$ is the solution of the the IVP (2).
- On the one hand, clearly the function $x \mapsto \frac{1}{x^2-2}$ is neither continuous at $\sqrt{2}$ nor at $-\sqrt{2}$. On the other hand, the interval should contain 0. Thus, the largest interval on which the solution to the IVP (2) is defined is $I = (-\sqrt{2}, \sqrt{2})$.

Exercise IV

On the one hand, using (3), we have
$$\begin{aligned} y' &= \frac{c\,e^t(1-c\,e^t) + c\,e^t(1+c\,e^t)}{(1-c\,e^t)^2} \\ &= \frac{2c\,e^t}{(1-c\,e^t)^2}. \end{aligned}$$
On the other hand, we have
$$\begin{aligned} y^2 - 1 &= \left(\frac{1+c\,e^t}{1-c\,e^t}\right)^2 - 1 \\ &= \frac{1 + c^2\,e^{2t} + 2c\,e^t - 1 - c^2\,e^{2t} + 2c\,e^t}{(1-c\,e^t)^2} \\ &= \frac{4c\,e^t}{(1-c\,e^t)^2} = 2y' \end{aligned}$$
That is
$$y' = \frac{1}{2}\,(y^2-1).$$
In other words, $y = \frac{1+c\,e^t}{1-c\,e^t}$, is a one-parameter family of solutions of the DE $y' = \frac{1}{2}(y^2-1)$.

In order to find a singular solution, we look for constant solutions (or equilibrium solutions). For this purpose, we let $y = k$, where k denotes an arbitrary constant. y is a solution of $y' = \frac{1}{2}(y^2-1)$ if and only if $0 = \frac{1}{2}(k^2-1)$ which means if and only if $k = \pm 1$. Thus clearly $k = 1$ and $k = -1$ are the constant solutions of the DE $y' = \frac{1}{2}(y^2-1)$. Now, observe that the solution $y = 1$ can be obtained from the one-parameter family of solutions $y = \frac{1+c\,e^t}{1-c\,e^t}$ by setting $c = 0$. However, there is no constant c such that $-1 = \frac{1+c\,e^t}{1-c\,e^t}$. Indeed,
$$\begin{aligned} -1 = \frac{1+c\,e^t}{1-c\,e^t} &\iff 1 + c\,e^t = -1 + c\,e^t \\ &\iff 1 = -1 \iff \text{absurd!} \end{aligned}$$
In summary, we demonstrated that $y = -1$ is a singular solution of the DE $y' = \frac{1}{2}\,(y^2-1)$.

Exercise V

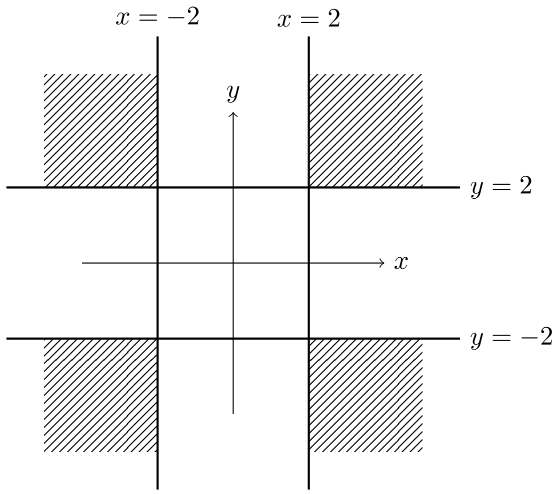
Let $y = x + \frac{c}{x}$, then
$$\begin{aligned} xy' + y &= x\left(1 - \frac{c}{x^2}\right) + x + \frac{c}{x} \\ &= x - \frac{c}{x} + x - \frac{c}{x} = 2x. \end{aligned}$$
Therefore, $y = x + \frac{c}{x}$ is a one-parameter family of solutions of the DE $xy' + y = 2x$.
To find a solution to the IVP (4), it is enough to solve for c the following equation
$$-2 - \frac{c}{2} = 1 \iff c = -6.$$
thus $y(x) = x - \frac{6}{x}$ is the desired solution. Clearly this function is not continuous at $x = 0$, and since the interval of validity should contain -2, we obtain that the interval of definition (or validity) is $I = (-\infty, 0)$.

Exercise VI

The constant solutions of the DE (5) are $y = k$ such that
$$k^2 + k = 0 \iff k(k+1) = 0 \iff k = 0 \,\text{or}\, k = -1.$$
Let us denote the constant solutions as $y_1 = 0$ and $y_2 = -1$ and check wether these solutions belong to the one-parameter family $y = \frac{c\,e^{-x}}{1-c\,e^{-x}}$. Clearly, if we set $c = 0$ in the latter expression, we obtain $y = 0 = y_1$. Thus y_1 is not a singular solution. Now, we check y_2, if y_2 is not a singular solution, then we should be able to find $c \in \mathbb{R}$ such that
$$\begin{aligned} -1 = \frac{c\,e^{-x}}{1-c\,e^{-x}} \iff -1 + c\,e^{-x} = c\,e^{-x} \iff -1 = 0 \\ \iff \text{absurd!} \end{aligned}$$
Thus, $y_2 = -1$ is a singular solution of the DE $y'' + y^2 + y = 0$.

Exercise VII

Thanks to the existence and uniqueness Theorem, the IVP (6) has a unique solution if f and $\frac{\partial f}{\partial y}$ are continuous on an open rectangle $\mathcal{R}$ containing (x_0, y_0) where
$$f(x,y) = \frac{\sqrt{y^2-4}}{\sqrt{x^2-4}}.$$
Clearly
$$\begin{aligned} \mathcal{R} &= \left\{(x,y) \in \mathbb{R}^2 \,|\, x^2 - 4 > 0 \,\text{and}\, y^2 - 4 > 0\right\} \\ &= \big\{(x,y) \in \mathbb{R}^2 \,|\, x \in (-\infty,-2) \cup (2,+\infty) \\ &\qquad \text{and}\, y \in (-\infty,-2) \cup (2,+\infty)\big\} \\ &= (-\infty,-2) \cup (2,+\infty) \,\times\, (-\infty,-2) \cup (2,+\infty). \end{aligned}$$



Exercise VIII

On the one hand, using (7), we have

$$\frac{dx}{dt} = \frac{18}{27}\,t\,(3t^2+c)^2 = \frac{2}{3}\,t\,(3t^2+c)^2.$$

On the other hand

$$\begin{aligned} 6t(x-1)^{\frac{2}{3}} &= 6t\,\left(\frac{1}{9}\,(3t^2+c)^2\right) \\ &= \frac{2}{3}\,t\,(3t^2+c)^2. \end{aligned}$$

This shows that x given by (7) defines a one-parameter family of solution of the DE (8). Next, constants solutions $x = k$ of the DE (8) are given by

$$0 = 6t(k-1)^{\frac{2}{3}}, \quad \text{for all} \quad t \in \mathbb{R}.$$

Clearly $x = k = 1$ is the only possible constant solution of the DE (8). Now, let us suppose that $x = 1$ is a member of the one-parameter family os solutions given by (7). Then, there exists $c \in \mathbb{R}$ such that for all $t \in \mathbb{R}$

$$1 = 1 + \frac{(3t^2+c)^3}{27} \iff 3t^2 + c = 0.$$

Therefore, we observe that c depends on t which contradicts the fact that it is a constant. All in all, we proved that $x = 1$ is a singular solution of the DE (8).

Exercise IX

One can easily check the fact that $2x^2 - y^2 - 2x = c$ defines a one-parameter family of solution of the DE (9). We want to find the particular solution satisfying the condition $y(1) = -2$. For this purpose, we simply equate this fact and obtain

$$c = -4.$$

Thus the solutions of the IVP

$$(9), \quad y(1) = -2.$$

Satisfy $2x^2 - y^2 - 2x = -4$. In particular,

$$y = \pm\sqrt{2x^2 - 2x + 4}.$$

But, as $y(1) = -2$, we obtain that the unique solution of the DE (9) is given by

$$y = -\sqrt{2x^2 - 2x + 4}.$$

Exercise X

If we let $f(x,y) = xy^{\frac{2}{3}}$, then

$$\frac{\partial f}{\partial y} = \frac{2}{3} x y^{-\frac{1}{3}},$$

which is defined and continuous for $y \neq 0$. As $\frac{\partial f}{\partial y}$ is not continuous on any open rectangle containing the point $(1,0)$, the existence and uniqueness Theorem does not guarantee that the IVP (10) has a unique solution.

Exercise XI

By implicit differentiation, we have

$$2xy^4 + 4x^2y'y^3 + 3x^2 = 0.$$

In particular, multiplying both sides of this equation by $1/x$, we get exactly the DE (11). Now, to claim that the DE (11) has a solution on $(0,3)$, we have to show that the relation $x^2y^4 + x^3 - 27 = 0$ has at least one solution defined on $(0,3)$. For this purpose, observe that this relation can be written as

$$y = \pm\left(\frac{27 - x^3}{x^2}\right)^{\frac{1}{4}},$$

which makes sense for all $x \in (0,3)$. Therefore, we conclude that the relation $x^2y^4 + x^3 - 27 = 0$ defines an implicit solution of the DE (11).

Remark 2.1. In order to be able to state that a relation $G(x,y) = 0$ defines an implicit solution to a given DE, we have to show that the equation $G(x,y) = 0$ has at least one solution. This is easily shown when we can extract the expression of the solution y from the equation $G(x,y)$ (and in this case, the solution is actually explicit and no longer implicit). When this is not possible, one has to resort to a result which is out of our MATH202 program, namely the implicit function Theorem. Roughly speaking, this Theorem tells you that if $G(x,y)$ is differentiable and its partial derivatives are continuous such that $\frac{\partial G}{\partial y} \neq 0$ for all (a,b) with $a \in I$ and b such that $G(a,b) = 0$, then the relation $G(x,y) = 0$ has at least one solution. Again this result is out of your program and we shall not ask you to use it. We mention it here just to be rigorous and for your mathematical culture.

Exercise XII

Let

$$f(x,y) = \frac{\sqrt{y-6x}}{x^2+1}.$$

Then, we have

$$\frac{\partial f}{\partial y} = \frac{1}{2(x^2+1)\sqrt{y-6x}}$$

Clearly f and $\frac{\partial f}{\partial y}$ are continuous on the region $\mathcal{R}$ given by

$$\mathcal{R} = \left\{(x,y) \in \mathbb{R}^2 \mid y > 6x\right\}.$$

Therefore, following the existence and uniqueness Theorem, the IVP (12) has a unique solution on an appropriate interval containing 5 if and only if $y(5) > 6 \times 5 = 30$, that is if and only if $b > 30$.