Additional Exercises
Math 202: Chapter 4

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This is a list of exercises with corrections. We do not pretend that this list covers entirely the whole material of the indicated sections, and we warmly recommend to not stop practicing after finishing these exercises. Never forget that knowledge is a treasure, but practice is the key to it.

1 Statement

Exercise I

Solve the DE

$$(1+x)y'' + xy' - y = 2(1+x)^2 e^x.$$

given that $y_1 = x$ and $y_2 = e^{-x}$ are linearly independent solutions of the associated Homogeneous DE on $I = (-1, +\infty)$.

Exercise II

Using the variation of parameters we find that a particular solution of the DE

$$y'' + y = f(t),$$

is

$$y_p(t) = u_1(t)\cos t + t \sin t,$$

where u_1 denotes a differentiable function of t. Find a candidate function f(t).

Exercise III

Without solving the DE, verify that

$$y = c_1 e^{2x} + c_2 e^{3x} + x^2,$$

is a general solution of the DE

$$y'' - 5y' + 6y = 6x^2 - 10x + 2.$$

Exercise IV

Show that the functions

$$f_1 = e^x, \quad f_2 = xe^x, \quad f_3 = x^2 e^x,$$

are linearly independent on $\mathbb{R}$.

Exercise V

Find a DE with general solution

$$y = c_1 e^{2x} + c_2 e^x \cos x + c_3 e^x \sin x + c_4 x e^x \cos x + c_5 x e^x \sin x + x^2.$$

Exercise VI

Consider the following differential equation

$$2t^2 + 3t - y = 0.$$

Given that $y_1 = t^{-1}$ is a solution of the DE,

1. find a suitable transformation to reduce the DE to a first order DE.
2. find all solutions to the DE.

2 Correction

Exercise I

The particular solution is given by

$$y_c = c_1 x + c_2 e^{-x}.$$

Now, we use the variation of constant method to find a particular solution of the form

$$y_p = u_1 y_1 + u_2 y_2.$$

It is rather easy to see that

$$W(x, e^{-x}) = \begin{vmatrix} x & e^{-x} \\ 1 & -e^{-x} \end{vmatrix} = -(x+1)e^{-x}.$$

Also, we have

$$W_1 = \begin{vmatrix} 0 & e^{-x} \\ 2(x+1)e^x & -e^{-x} \end{vmatrix} = -2(x+1),$$

and

$$W_2 = \begin{vmatrix} x & 0 \\ 1 & 2(x+1)e^x \end{vmatrix} = 2x(x+1)e^x.$$

Therefore, we have

$$u_1(x) = \int \frac{-2(x+1)}{-(x+1)e^{-x}}\,dx = 2e^x + c,$$

and

$$\begin{aligned} u_2(x) &= \int \frac{2x(x+1)e^x}{-(x+1)e^{-x}}\,dx \\ &= -2\int xe^{2x}\,dx \\ &= (\frac{1}{2} - x)e^{2x} + c \end{aligned}$$

We take $c = 0$ in both expressions and find out that

$$y_p = 2xe^x + (\frac{1}{2} - x)e^{2x}e^{-x} = (x + \frac{1}{2})e^x.$$

That is Thus the general solution of the DE is given by

$$y = c_1 x + c_2 e^{-x} + (x + \frac{1}{2})e^x.$$

Exercise II

The complementary solution is given by

$$y_c = c_1 \cos t + c_2 \sin t.$$

variation of parameter provides a particular solution

$$y_p(t) = u_1(t)\cos t + u_2(t)\sin t.$$

One may take $u_2(t) = t$, thus $u_2'(t) = 1$. That is

$$u_2'(t) = 1 = \frac{\begin{vmatrix} \cos t & 0 \\ -\sin t & f(t) \end{vmatrix}}{\begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix}} = f(t)\cos t.$$

Thus, we get

$$f(t) = \frac{1}{\cos t}.$$

Now, we have

$$\begin{aligned} u_2'(t) = \frac{\begin{vmatrix} 0 & \sin t \\ f(t) & \cos t \end{vmatrix}}{\begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix}} &= -f(t)\sin t \\ &= -\frac{\sin t}{\cos t}. \end{aligned}$$

So, we take $u_1(t) = \ln|\cos t|$, and consequently

$$y_p(t) = \ln|\cos t|\cos t + t\sin t.$$

Eventually, teh general solution of the DE is given by

$$y = c_1 \cos t + c_2 \sin t \ln|\cos t|\cos t + t\sin t.$$

Exercise III

Clearly $y = x^2$ is a particular solution of the DE and $y_c = c_1 e^{2x} + c_2 e^{3x}$ is the complementary solution of the DE. If we let $y_1 = e^{2x}$ and $y_2 = e^{3x}$, one can see clearly that their are solutions of the homogeneous differential equation and they are linearly independent. Indeed, we have $y_1' = 2y_1$ and $y_1'' = 4y_1$. Thus

$$y_1'' - 5y_1' + 6y_1 = (4 - 10 + 6)y_1 = 0.$$

Also, $y_2' = 3y_1$ and $y_1'' = 9y_1$ so that

$$y_2'' - 5y_2' + 6y_2 = (9 - 15 + 6)y_2 = 0.$$

Also,

$$W(y_1, y_2) = \begin{vmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{vmatrix} = e^{5x} \neq 0.$$

Thus $\{y_1, y_2\}$ is a fundamental set of solutions of the DE. Eventually, it remains to check that x^2 is a particular solution of the DE which is true since $2 - 5(2x) + 6(x^2) = 6x^2 - 10x + 2$. We conclude that

$$y = c_1 e^{2x} + c_2 e^{3x} + x^2,$$

is the general solution of the DE.

Exercise IV

It suffices to calculate the wronskian

$$\begin{aligned} W(f_1, f_2, f_3) &= \begin{vmatrix} e^x & xe^x & x^2e^x \\ e^x & (x+1)e^x & (2x+x^2)e^x \\ e^x & (x+2)e^x & (2+4x+x^2)e^x \end{vmatrix} \\ &= e^{3x}\begin{vmatrix} 1 & x & x^2 \\ 1 & (x+1) & (2x+x^2) \\ 1 & (x+2) & (2+4x+x^2) \end{vmatrix} \\ &= e^{3x}\begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 2 & 2+4x \end{vmatrix} \\ &= (2 + 4x - 4x)e^{3x} = 2e^{3x} \neq 0. \end{aligned}$$

This leads to the desired result.

Exercise V

Clearly the first part of the solution is the complementary solution. That is a solution of homogeneous linear differential equation whose characteristic equation has as roots

- 2 as a simple root
- $1 + i$ and therefore its complex conjugate $1 - i$ as solutions of multiplicity 2 each.

We Deduce that the characteristic equation should be of the form

$$(r-2)(r-1-i)^2(r-1+i)^2 = 0.$$

developing this equation, we obtain

$$r^5 - 6r^4 + 16r^3 - 16r^2 + 20r - 8 = 0$$

That is our homogenous differential equation is given by

$$y^{(5)} - 6y^{(4)} + 16y^{(3)} - 16y'' + 20y' - 8y = 0.$$

Since we have x^2 as part of the general solution, we deduce that our linear differential equation is non homogenous and we have to find the right hand term. For this, we have just to plug $y = x^2$ in

$$y^{(5)} - 6y^{(4)} + 16y^{(3)} - 16y'' + 20y' - 8y$$

and take the result as the right hand term. Doing this, we obtain

$$\begin{aligned}(x^2)^{(5)} - 6(x^2)^{(4)} + 16(x^2)^{(3)} - 16(x^2)'' + 20(x^2)' - 8(x^2)\\ = -8x^2 + 40x - 32.\end{aligned}$$

All in all, our fifth order linear non homogeneous linear differential equation is given by

$$y^{(5)} - 6y^{(4)} + 16y^{(3)} - 16y'' + 20y' - 8y = -8x^2 + 40x - 32.$$

Exercise VI

1. We let $y = uy_1 = ut^{-1}$. Then

$$y' = -t^{-2}u + t^{-1}u',$$

and

$$y'' = 2t^{-3}u - 2t^{-2}u' + t^{-1}u''.$$

Substituting in the DE, we obtain after easy calculation

$$2tu'' - u' = 0.$$

Now, setting $v = u'$, we get the first order DE

$$2tv' - v = 0.$$

2. the DE $2tv' - v = 0$ can be easily solved since it is a separable DE and we obtain

$$v(t) = k_1\sqrt{t}, \quad k \in \mathbb{R}.$$

integrating this we obtain u. That is

$$u = k_1\frac{2}{3}t^{\frac{3}{2}} + k_2.$$

Therefore,

$$u = \alpha t^{\frac{3}{2}} + \beta, \quad \alpha, \beta \in \mathbb{R}.$$

It follows that

$$y = \frac{u}{t} = \alpha t^{\frac{1}{2}} + \beta t^{-1}, \quad \alpha, \beta \in \mathbb{R},$$

is the general solution of the original DE.