Additional Exercises
Math 202: Sections 4.1, 4.2 and 4.3

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This is a list of exercises with corrections. We do not pretend that this list covers entirely the whole material of the indicated sections, and we warmly recommend to not stop practicing after finishing these exercises. Never forget that knowledge is a treasure, but practice is the key to it.

1 Statement

Exercise I

Show that $y_1(x) = \cos\ln x$ and $y_2 = \sin\ln x$ are linearly independent on $I = (0, +\infty)$.

Exercise II

Verify that the set of functions $\{x, x^2, \frac{1}{x}\}$ forms a fundamental set of solutions of the differential equation

$$x^3y''' + x^2y'' - 2xy' + 2y = 0, \quad \text{on} \quad (0, +\infty). \tag{1}$$

Find the general solution of the differential equation (1).

Exercise III

Given that $y_{p_1} = e^{3x}$ is a particular solution of the differential equation

$$y'' - 3y' + 2y = 2e^{3x}, \tag{2}$$

and that $y_{p_2} = x^2 + 3x + 3$ is a particular solution of the differential equation

$$y'' - 3y' + 2y = 2x^2 - 1, \tag{3}$$

find a particular solution of the differential equation

$$y'' - 3y' + 2y = 7e^{3x} + 6x^2 - 3. \tag{4}$$

Exercise IV

Given that $y_1(x) = \frac{1}{x}$ is a solution of the differential equation

$$x^2y'' + 3xy' + y = 0, \quad x > 0, \tag{5}$$

use the method of reduction of order to find a second solution of (5) that is linearly independent of y_1.

Exercise V

Find the general solution of the differential equation

$$y^{(5)} + 3y^{(4)} - 5y''' + 17y'' - 36y' + 20y = 0. \tag{6}$$

Exercise VI

Find the general solution of the differential equation

$$y^{(4)} - 7y'' - 18y = 0. \tag{7}$$

Exercise VII

Find the general solution of the differential equation

$$y''' + 6y'' + y' - 34y = 0,$$

knowing that $e^{-4x}\cos x$ is a solution.

Exercise VIII

Find the general solution of the differential equation

$$2y''' + 7y'' + 4y' - 4y = 0.$$

Observe that $\frac{1}{2}$ is a root of the associated auxiliary equation.

2 Correction

Exercise I

The Wronskian is given by

$$W(y_1,y_2)(x) = \begin{vmatrix} \cos\ln x & \sin\ln x \\ -\frac{\sin\ln x}{x} & \frac{\cos\ln x}{x} \end{vmatrix} = \frac{1}{x}\left(\cos^2\ln x + \sin^2\ln x\right) = \frac{1}{x} \neq 0,$$

for all $x \in (0,+\infty)$. We infer that y_1 and y_2 are linearly independent on $I = (0,+\infty)$.

Exercise II

One may check easily that $y_1 = x, y_2 = x^2$ and $y_3 = \frac{1}{x}$ are solutions of the differential equation (1). Indeed, $y_1' = 1$ and its higher derivatives are 0 and owe have clearly

$$x^3y_1''' + x^2y_1'' - 2xy_1' + 2y_1 = -2x + 2x = 0.$$

Also, we have $y_2' = 2x, y_2'' = 2$ and $y_2''' = 0$. Thus, we have

$$x^3y_2''' + x^2y_2'' - 2xy_2' + 2y_2 = 2x^2 - 4x^2 + 2x^2 = 0$$

Eventually, we have $y_3' = -\frac{1}{x^2}, y_3'' = \frac{2}{x^3}$ and $y_3''' = -\frac{6}{x^4}$ so that

$$x^3y_3''' + x^2y_3'' - 2xy_3' + 2y_3 = -\frac{6}{x} + \frac{2}{x} + \frac{2}{x} + \frac{2}{x} = 0.$$

Now, we have

$$\begin{aligned} W(y_1,y_2,y_3)(x) &= \begin{vmatrix} x & x^2 & \frac{1}{x} \\ 1 & 2x & -\frac{1}{x^2} \\ 0 & 2 & \frac{2}{x^3} \end{vmatrix} \\ &= x\begin{vmatrix} 2x & -\frac{1}{x^2} \\ 2 & \frac{2}{x^3} \end{vmatrix} - \begin{vmatrix} x^2 & \frac{1}{x} \\ 2 & \frac{2}{x^3} \end{vmatrix} \\ &= x\left(\frac{4}{x^2} + \frac{2}{x^2}\right) - \left(\frac{2}{x} - \frac{2}{x}\right) \\ &= \frac{6}{x} \neq 0, \end{aligned}$$

for all $x \in (0,+\infty)$. Therefore, $\{y_1,y_2,y_3\}$ is a fundamental set of solutions of the differential equation (1). The general solution is given by

$$y = c_1x + c_2x^2 + \frac{c_3}{x}, \quad c_1,c_2,c_3 \in \mathbb{R}.$$

Exercise III

Let L de the linear differential operator given by

$$L = D^2 - 3D + 2.$$

Then, we havqe

$$Ly_{p_1} = 2e^{3x}, \quad \text{and} \quad Ly_{p_2} = 2x^2 - 1.$$

Now, observe that we have

$$7e^{3x} + 6x^2 - 3 = \frac{7}{2}(2e^{3x}) + 3(2x^2 - 1).$$

Therefore,

$$\begin{aligned} 7e^{3x} + 6x^2 - 3 &= \frac{7}{2}Ly_{p_1} + 3Ly_{p_2} \\ &= L\,\frac{7}{2}y_{p_1} + L\,3y_{p_2} \\ &= L\,(\frac{7}{2}y_{p_1} + 3y_{p_2}). \end{aligned}$$

Thus, $y_p = \frac{7}{2}y_{p_1} + 3y_{p_2}$ is a particular solution of the differential equation (4). In particular

$$y_p = \frac{7}{2}e^{3x} + 3x^2 + 9x + 9.$$

Exercise IV

The standard form of the differential equation (5) is

$$y'' + \frac{3}{x}\,y' + \frac{1}{x^2}\,y = 0.$$

By reduction of order, we let

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{-\int \frac{3}{x}\,dx}}{y_1^2}\,dx \\ &= y_1 \int \frac{\frac{1}{x^3}}{y_1^2}\,dx \\ &= y_1 \int \frac{dx}{x} \\ &= \frac{\ln x}{x}. \end{aligned}$$

Thus, y_2 is a second solution of the differential equation (5) which is linearly independent of y_1. The general solution of the differential equation (5) is then given by

$$y = \frac{c_1}{x} + c_2\,\frac{\ln x}{x}, \quad c_1,c_2 \in \mathbb{R}.$$

Exercise V

The auxiliary (or characteristic) equation associated to the differential equation (6) is

$$\begin{aligned} & r^5 + 3r^4 - 5r^3 + 17r^2 - 36r + 20 = 0 \\ \Leftrightarrow\ & (r-1)(r^4 + 4r^3 - r^2 + 16r + 20) = 0 \\ \Leftrightarrow\ & (r-1)^2(r^3 + 5r^2 + 4r + 20) = 0 \\ \Leftrightarrow\ & (r-1)^2(r^2(r+5) + 4(r+5)) = 0 \\ \Leftrightarrow\ & (r-1)^2(r+5)(r^2+4) = 0 \\ \Leftrightarrow\ & (r-1)^2(r+5)(r+2i)(r-2i) = 0 \end{aligned}$$

It follows clearly that the general solution of the differential equation (6) is given by

$$c_1e^x + c_2xe^x + c_3e^{-5x} + c_4\cos(2x) + c_5\sin(2x),$$

where

$$c_k \in \mathbb{R}, \quad k = 1, \ldots, 5.$$

Exercise VI

The auxiliary equation of the differential equation (7) is

$$r^4 - 7r^2 - 18 = 0.$$

Clearly, this equation is equivalent to

$$(r^2+2)(r^2-9) = 0 \Leftrightarrow (r-\sqrt{2}i)(r+\sqrt{2}i)(r-3)(r+3) = 0$$

The general solution of the differential equation (7) is then given by

$$y = c_1 e^{3x} + c_2 e^{-3x} + c_3 \cos(\sqrt{2}\,x) + c_4 \sin(\sqrt{2}\,x),$$

where

$$c_1, c_2, c_3, c_4 \in \mathbb{R}.$$

Exercise VII

Since $e^{-4x} \cos x$ is a solution, than we deduce that $-4+i$ is a solution of the characteristic equation and therefore its conjugate $-4-i$ is as well a solution. The characteristic equation can be written then

$$\begin{aligned} 0 &= r^3 + 6r^2 + r - 34 \\ &= (r+4-i)(r+4+i)(r-2). \end{aligned}$$

THus, the third root is $r = 2$, and therefore the general solution of the differential equation is given by

$$y = c_1 e^{-4x} \cos x + c_2 e^{-4x} \sin x + c_3 e^{2x},$$

where

$$c_1, c_2, c_3 \in \mathbb{R}.$$

Exercise VIII

The characteristic equation of the differential equation is given by

$$2r^3 + 7r^2 + 4r - 4 = 0.$$

As $r = \frac{1}{2}$ is a solution, we have clearly

$$\begin{aligned} 0 &= 2r^3 + 7r^2 + 4r - 4 \\ &= (r - \frac{1}{2})(2r^2 + \alpha r + \beta). \end{aligned}$$

Developing this expression and using identification, we end up with the fact that $\alpha = 8$ and $\beta = 8$. Thus, the solutions of the characteristic equation are $r = \frac{1}{2}$ and $r = -2$ which is of multiplicity 2. Eventually, the general solution is then given by

$$y = c_1 e^{\frac{x}{2}} + c_2 e^{2x} + c_3 x e^{2x},$$

where

$$c_1, c_2, c_3 \in \mathbb{R}.$$

3 Remark on Reduction of Order

Consider the differential equation

$$y'' + p(x)y' + q(x)y = f(x), \tag{8}$$

where $p(x), q(x)$ and $f(x)$ are continuous on some open interval I.

Assume that $y_1(x)$ is a solution of the homogeneous equation associated to (8) such that $y_1(x) \neq 0$ for all $x \in I$. That is

$$y_1'' + p(x)y_1' + q(x)y_1 = 0, \quad \forall\, x \in I$$

If y is a solution of the differential equation (8), then we let

$$y = u\, y_1.$$

Therefore, u satisfies

$$u'' y_1 + (2y_1' + p(x)y_1)u' = f(x).$$

Letting $v = u'$, we get

$$v' + (2\frac{y_1'}{y_1} + p(x))v = \frac{f(x)}{y_1}.$$

This is a first order linear differential equation. Its integrating factor is

$$\begin{aligned} e^{\int \left(2\frac{y_1'}{y_1} + p(x)\right)\, dx} &= e^{2\ln|y_1|}\, e^{\int p(x)dx} \\ &= y_1^2\, e^{\int p(x)dx}. \end{aligned}$$

Now, multiplying the differential equation by this integrating factor we get

$$\frac{d}{dx}\left(y_1^2\, e^{\int p(x)dx}\, v\right) = y_1\, e^{\int p(x)dx}\, f(x).$$

Thus, we have

$$v = y_1^{-2}\, e^{-\int p(x)dx} \int y_1\, e^{\int p(x)dx}\, f(x)\, dx$$

Eventually, to get u, it suffices to integrate v.

Exercise IX

The function e^x is a solution of the differential equation

$$xy'' - 2y' + (2-x)y = x^3, \quad x \in (0, +\infty) \tag{9}$$

Use the method of reduction of order to find the general solution of the differential equation (9).

Solution of Exercise IX

The normal form of the differential equation (9) reads

$$y'' - \frac{2}{x}y' + \left(\frac{2}{x} - 1\right) y = x^2.$$

Now, we set $y = uy_1 = ue^x$ and $v = u'$ and obtain

$$v' + \left(2\frac{e^x}{e^x} - \frac{2}{x}\right) v = x^2 e^{-x}.$$

That is

$$v' + 2\left(1 - \frac{1}{x}\right) v = x^2 e^{-x}.$$

This is a first order linear differential equation. Its integrating factor is

$$e^{2\int\left(1-\frac{1}{x}\right)dx} = \frac{e^{2x}}{x^2}.$$

Now, multiplying the differential equation on both sides by this integrating factor, we get

$$\frac{d}{dx}\left(\frac{e^{2x}}{x^2}\, v\right) = \frac{e^{2x}}{x^2}\frac{x^2}{e^x} = e^x$$

Thus, we have

$$v = \frac{x^2}{e^{2x}} \int e^x \, dx = \frac{x^2}{e^x}\left(1 + c_1 e^{-x}\right),$$

where $c_1 \in \mathbb{R}$. Now, we go back and find u. We have

$$\begin{aligned} u(x) &= \int \frac{x^2}{e^x}\left(1 + c_1 e^{-x}\right) \, dx \\ &= -(x^2 + 2x + 2)\, e^{-x} + c_2 \\ &- \frac{c_1}{2}\left(x^2 + x + \frac{1}{2}\right) e^{-2x}. \end{aligned}$$

Now, we go back to find y. We have

$$\begin{aligned} y(x) = uy_1 &= -(x^2 + 2x + 2) + c_2 e^x \\ &- \frac{c_1}{2}(x^2 + x + \frac{1}{2})e^{-x}, \end{aligned}$$

where

$$c_1, c_2 \in \mathbb{R},$$

which is a second solution of the differential equation (9).