Additional Exercises
Math 202: Sections 2.4 & 2.5

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This is a list of exercises with corrections. We do not pretend that this list covers entirely the whole material of the indicated sections, and we warmly recommend to not stop practicing after finishing these exercises. Never forget that knowledge is a treasure, but practice is the key to it.

1 Statement

Exercise I

Find a suitable substitution that transforms the DE

$$(x^2 + y^2)y' = xy, \tag{1}$$

into a separable DE (do not solve it).

Exercise II

Solve the initial value problem

$$2xy' + y = 6x, \quad y(4) = 20, \tag{2}$$

on the interval $I = (0, +\infty)$

Exercise III

Solve the DE

$$(y \cos x + 2xe^y)\, dx + (\sin x + x^2 e^y - 1)\, dy = 0. \tag{3}$$

Exercise IV

Consider the DE

$$(3xy + y^2) + (x^2 + xy)y' = 0. \tag{4}$$

1. Show that the DE (4) is not exact.
2. Find an integrating factor, that converts the DE (4) to an exact DE.

Exercise V

1. Transform the DE
$$xy' = \frac{y^2}{x} + y,$$
into a separable DE.
2. Transform the DE
$$y' + \frac{1}{x}y = 3x^2y^3, \tag{5}$$
into a linear DE.

Exercise VI

Consider the DE

$$y' = \frac{x + y - 1}{x - y - 2}. \tag{6}$$

1. For which values of a and b the change of variables $x = t + a$, and $y = z + b$ transforms the DE (6) into
$$\frac{dz}{dt} = \frac{t + z}{t - z}. \tag{7}$$
2. Find a suitable substitution transforming the DE (7) into a separable DE.

Exercise VII

Use a suitable substitution to transform the DE

$$\frac{dy}{dx} = (x + y)\, e^{x+y}, \tag{8}$$

into a separable DE.

Exercise VIII

Use an appropriate substitution to reduce the DE

$$\frac{dy}{dx} = 2 + \sqrt{y - 2x + 3}. \tag{9}$$

2 Correction

Exercise I

The DE (1) can be rephrased as

$$M(x,y)\,dx+N(x,y)dy=0,$$

where

$$M(x,y)=-xy,\quad \text{and} \quad N(x,y)=x^2+y^2.$$

Clearly, for all $\lambda\in\mathbb{R}$, we have

$$M(\lambda x,\lambda y)=\lambda^2(x^2+y^2)=\lambda^2 M(x,y),$$

and

$$N(\lambda x,\lambda y)=-\lambda^2 xy=\lambda^2 N(x,y).$$

Thus $M$and N are homogeneous functions of degree 2. Therefore

$$\frac{dy}{dx}=f(\frac{y}{x}),$$

where

$$f(\frac{y}{x})=-\frac{M(1,\frac{y}{x})}{N(1,\frac{y}{x})}=\frac{\frac{y}{x}}{1+\left(\frac{y}{x}\right)^2}.$$

Now, we use the substitution

$$u=\frac{y}{x}\Longleftrightarrow y=xu\Longleftrightarrow \frac{dy}{dx}=u+x\frac{du}{dx}.$$

Thus, we have

$$u+x\frac{du}{dx}=f(u)=\frac{u}{1+u^2}.$$

In particular

$$\frac{du}{dx}=\frac{1}{x}\left(\frac{u}{1+u^2}-u\right),$$

or

$$\frac{du}{dx}=-\frac{1}{x}\left(\frac{u^2}{1+u^2}\right).$$

This is clearly a separable DE.

Exercise II

The DE in the IVP (2) is clearly linear first order . Its normal (or standard) form is

$$y'+\frac{1}{2x}y=3. \tag{10}$$

Its integrating factor is given by

$$u(x)=\exp\left\{\int\frac{1}{2x}\,dx\right\}=\sqrt{x}.$$

Multiplying both sides of the DE (10) by $\sqrt{x}$, we obtain

$$\left(\sqrt{x}y\right)'=3\sqrt{x}.$$

Integrating the latter inequality, we obtain

$$\sqrt{x}y=2\,x^{\frac{3}{2}}+c.$$

In particular

$$y=2x+\frac{c}{\sqrt{x}}.$$

Using the initial condition in the IVP(2), we obtain

$$20=8+\frac{c}{2}\Longleftrightarrow c=24.$$

Eventually, the function

$$y=2x+\frac{24}{\sqrt{x}},$$

is a solution of the IVP (2) on $(0,+\infty)$.

Exercise III

We start by checking wether the DE (3) is exact or not. For this purpose, we write

$$\frac{\partial}{\partial y}(y\,\cos x+2xe^y)=\cos x+2xe^y.$$

and

$$\frac{\partial}{\partial x}(\sin x+x^2e^y-1)=\cos x+2xe^y.$$

We infer that the DE (3) is exact. Therefore, there exist a potential function $\varphi(x,y)$ such that

$$\begin{cases}\frac{\partial\varphi}{\partial x}=y\,\cos x+2xe^y,\\[2mm] \frac{\partial\varphi}{\partial y}=\sin x+x^2e^y-1.\end{cases}$$

The first equation leads to

$$\varphi(x,y)=y\sin x+x^2e^y+g(y).$$

That is,

$$\begin{aligned}\frac{\partial\varphi}{\partial y}&=\sin x+x^2e^y+g'(y)\\ &=\sin x+x^2e^y-1.\end{aligned}$$

We get easily

$$g'(y)=-1\Longleftrightarrow g(y)=-y+c.$$

Since, any integrating factor will do, we set here $c=0$. Therefore, the potential function is given by

$$\varphi(x,y)=y\sin x+x^2e^y-y.$$

Eventually, the relation

$$y\sin x+x^2e^y-y=c,\quad \text{for all} \quad c\in\mathbb{R},$$

defines a set of solutions of the DE (3).

Exercise IV

1. We rephrase the DE (4) as follows

$$M(x,y)dx + N(x,y)dy = 0,$$

where

$$M(x,y) = 3xy + y^2, \quad \text{and} \quad N(x,y) = x^2 + xy.$$

Now we have

$$\frac{\partial M}{\partial y} = 3x + 2y,$$

and

$$\frac{\partial N}{\partial x} = 2x + y.$$

Clearly,

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}.$$

Thus, the DE (4) is not exact.

2. Observe that

$$\begin{aligned}\frac{M_y - N_x}{N} &= \frac{3x + 2y - 2x - y}{x^2 + xy} \\ &= \frac{x + y}{x(x+y)} = \frac{1}{x}.\end{aligned}$$

Therefore, an integrating factor of the DE (4) is given by

$$u(x) = \exp\left\{\int \frac{dx}{x}\right\} = |x|.$$

The choice $u(x) = x$ will do. Now, we multiply both sides of the DE (4) by x, we get

$$(3x^2y + xy^2)dx + (x^3 + x^2y)dy = 0.$$

Therefore, this DE is exact.

Exercise V

1. We rephrase our DE as follows

$$M(x,y)dx + N(x,y)dy = 0,$$

where

$$M(x,y) = \frac{y^2}{x} + y, \quad \text{and} \quad N(x,y) = -x.$$

M and N are clearly homogenous of degree 1. Therefore, the DE at hand is homogenous of degree 1. Thus, we can write is as

$$y' = -\frac{M(1,\frac{y}{x})}{N(1,\frac{y}{x})} = -\frac{\frac{y^2}{x^2} + \frac{y}{x}}{-1}.$$

Let

$$u = \frac{y}{x} \iff y = xu \iff \frac{dy}{dx} = u + x\frac{du}{dx}.$$

That is

$$\frac{du}{dx} = \frac{(u^2 + u) - u}{x} = \frac{u^2}{x}.$$

This is clearly a separable DE.

[2.] The De (5) is a Bernoulli. We multiply it by $\frac{1}{y^3}$ (Red Triangle here, we assume $y \neq 0$) to get

$$y^{-3}y' + \frac{1}{x}y^{-2} = 3x^2.$$

Next, we use the substitution

$$v = y^{-2} \iff v' = -2y'y^{-3} \iff y'y^{-3} = -\frac{1}{2}v'.$$

The DE can be written now as

$$-\frac{1}{2}v' + \frac{1}{x}v = 3x^2.$$

Which is a linear first order differential equation that can be written in its normal form as

$$v' - \frac{2}{x}v = -6x^2.$$

An integrating factor of this De is

$$u(x) = \exp\left\{-2\int \frac{dx}{x}\right\}.$$

For instance, we choose

$$u(x) = \frac{1}{x^2}.$$

That is, we have

$$\left(\frac{v}{x^2}\right)' = -6.$$

Integrating, we obtain

$$v = -6x^3 + cx^2 = x^2(c - 6x), \quad c \in \mathbb{R}.$$

That is

$$y = \frac{\pm 1}{\sqrt{x^2(c-6x)}}.$$

Of course, do not forget that $y = 0$ is a trivial (and singular) solution of the DE (5).

Exercise VI

1. Using the chain rule, we have

$$\frac{dz}{dt} = \frac{dz}{dy}\frac{dy}{dx}\frac{dx}{dt} = 1 \times \frac{dy}{dx} \times 1 = \frac{dy}{dx}.$$

That is

$$\frac{dz}{dt} = \frac{t + z + a + b - 1}{t - z + a - b - 2}$$

Therefore we look for a and b such that

$$\begin{cases} a + b - 1 = 0, \\ a - b - 2 = 0. \end{cases}$$

It is rather easy to see that $a = \frac{3}{2}$ and $b = -\frac{1}{2}$.

2. The DE (7) can be rephrased as follows

$$M(t,z)dt + N(t,z)dz = 0,$$

where

$$M(t,z) = t + z, \quad \text{and} \quad N(t,z) = z - t.$$

$$M(\lambda t, \lambda z) = \lambda(t+z) = \lambda M(t,z),$$

and

$$N(\lambda t, \lambda z) = \lambda(z-t) = \lambda N(t,z).$$

Thus $M$and N are homogeneous functions of degree 1. Therefore

$$\frac{dz}{dt} = f(\frac{z}{t}),$$

where

$$f(\frac{z}{t}) = -\frac{M(1,\frac{z}{t})}{N(1,\frac{z}{t})} = -\frac{1+\frac{z}{t}}{\frac{z}{t}-1}.$$

Now, we use the substitution

$$u = \frac{z}{t} \iff z = tu \iff \frac{dz}{dt} = u + t\frac{du}{dt}.$$

Thus, we have

$$u + t\frac{du}{dt} = f(u) = \frac{u+1}{u-1}.$$

That is

$$\frac{du}{dt} = f(u) = \frac{1}{t}\left(\frac{u+1}{1-u} - u\right) = \frac{1}{t}\frac{1+u^2}{1-u},$$

which is clearly a separable DE.

Exercise VII

We observe that if we set $u = x + y$, then

$$\begin{aligned}\frac{du}{dx} &= 1 + \frac{dy}{dx} \\ &= 1 + (x+y)\,e^{x+y} \\ &= 1 + ue^u.\end{aligned}$$

This is clearly a separable DE.

Exercise VIII

If we let $u = y - 2x + 3$, then

$$\begin{aligned}\frac{du}{dx} &= \frac{dy}{dx} - 2 \\ &= 2 + \sqrt{y-2x+3} - 2 \\ &= \sqrt{u}. \qquad (11)\end{aligned}$$

Thus, we obtained a separable DE. Integrating, we get (Red Triangle here, we assume $u \neq 0$)

$$\int \frac{du}{\sqrt{u}} = x + c, \quad c \in \mathbb{R}$$

which is equivalent to

$$2\sqrt{u} = x + c.$$

That is

$$u = \frac{1}{2}(x+c)^2.$$

Eventually, the solutions of the DE (9) are given by

$$y - 2x + 3 = \frac{1}{2}(x+c)^2, \quad c \in \mathbb{R}$$

Do not forget that $u = 0$ is a solution of (11). That is $y = 2x - 3$ is a solution of (9), which is a singular solution.