Additional Exercises
Math 202: Sections 2.2 & 2.3

Dr. Othman Echi and Dr. Saber Trabelsi

Department of Mathematics & Statistics, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia.

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This is a list of exercises with corrections. We do not pretend that this list covers entirely the whole material of the indicated sections, and we warmly recommend to not stop practicing after finishing these exercises. Never forget that knowledge is a treasure, but practice is the key to it.

1 Statement

Exercise I

Solve the IVP

$$\frac{dy}{dx} = \frac{xy - 3x - 2y + 6}{xy - 2x - 3y + 6}, \quad y(0) = 1. \tag{1}$$

Exercise II

Find an implicit solution of the IVP

$$y' = \frac{\sin x}{\cos y} e^{-\sin y - \cos x}, \quad y(\frac{\pi}{2}) = 0. \tag{2}$$

Exercise III

Solve the IVP

$$x\, e^{2x + \cos y}\, dx + \sin y\, dy = 0, \quad y(0) = \frac{\pi}{2}. \tag{3}$$

Exercise IV

Solve the IVP

$$\sin x\, dx + 2y \cos x\, dy = 0, \quad y(0) = 1. \tag{4}$$

Exercise V

Solve the linear DE

$$(y+1)\frac{dy}{dx} + (y+2)\, x = 2ye^{-y}. \tag{5}$$

Exercise VI

Show that the DE

$$\frac{dy}{dx} = \frac{y}{ye^y - 2x}, \tag{6}$$

is linear in x and find its solutions.

Exercise VII

Solve the DE

$$t\frac{dy}{dx} + y - t^4 \ln t = 0, \tag{7}$$

and find its interval of validity.

Exercise VIII

Solve the initial value problem

$$x^2(x-2)\frac{dy}{dx} + x(x-2)y = 2, \quad y(1) = 1, \tag{8}$$

and give the interval of definition of the solution.

Exercise IX

Find the solution of the following initial value problem

$$(1+x^2)\frac{dy}{dx} + 2xy = f(x), \quad y(0) = 0, \tag{9}$$

on $\mathbb{R}$, where

$$f(x) = \begin{cases} x & \text{if } \quad x < 1, \\ 2x - 1 & \text{if } \quad x \geq 1. \end{cases}$$

2 Correction

Exercise I

Observe that

$$\begin{aligned} xy - 3x - 2y + 6 &= x(y-3) - 2(y-3) \\ &= (x-2)(y-3), \end{aligned}$$

and

$$\begin{aligned} xy - 2x - 3y + 6 &= x(y-2) - 3(y-2) \\ &= (x-3)(y-2). \end{aligned}$$

Thus, the DE in the IVP (1) is equivalent to

$$\frac{dy}{dx} = \frac{x-2}{x-3}\,\frac{y-3}{y-2},$$

which is clearly separable. Therefore, separating variables, we get

$$\frac{x-2}{x-3}\,dx = \frac{y-2}{y-3}\,dy.$$

Integrating, we have

$$\begin{aligned} &\int \frac{x-2}{x-3}\,dx = \int \frac{y-2}{y-3}\,dy + c \\ \Leftrightarrow &\int \frac{x-3+1}{x-3}\,dx = \int \frac{y-3+1}{y-3}\,dy + c \\ \Leftrightarrow &\int \left(1 + \frac{1}{x-3}\right)\,dx = \int \left(1 + \frac{1}{y-3}\right)\,dy + c \\ \Leftrightarrow &\; x + \ln|x-3| = y + \ln|y-3| + c. \end{aligned}$$

The initial condition $y(0) = 1$ leads to

$$\ln 3 = 1 + \ln 2 + c \Leftrightarrow c = \ln\frac{3}{2} - 1.$$

Eventually, the solution of the IVP (1) is given by the implicit relation

$$x + \ln|x-3| = y + \ln|y-3| + \ln\frac{3}{2} - 1.$$

Exercise II

The DE in the IVP (2) is equivalent to

$$y' = \sin x\, e^{-\cos x}\,\frac{e^{-\sin y}}{\cos y},$$

which is clearly separable. Separating the variables, we obtain

$$\cos y\, e^{\sin y}\,dy = \sin x\, e^{-\cos x}\,dx.$$

Integrating this equality, we get

$$e^{\sin y} = e^{-\cos x} + c.$$

Now, the initial condition leads clearly to

$$c = 0.$$

It follows that the solution of the IVP (2) is given by the relation

$$e^{\sin y} = e^{-\cos x} \Leftrightarrow \sin y = -\cos x \Leftrightarrow y = \sin^{-1}(-\cos x).$$

The interval of validity of the solution is $(0, \pi)$ which is the largest interval containing $\frac{\pi}{2}$ such that $y(x)$ is differentiable on and $\cos y \neq 0$ (check that y is not differentiable at 0 and π. Also, observe that at $x = 0$ and $x = \pi$, one has $\cos(y) = 0$).

Exercise III

The DE in the IVP (3) is equivalent to

$$xe^{2x}\,dx = -\sin y e^{-\cos y}\,dy,$$

which is clearly separable. Integrating this equality, we obtain

$$\int xe^{2x}\,dx = -\int \sin y e^{-\cos y}\,dy + c.$$

Now, using integration by parts, we can write

$$\begin{aligned} \int xe^{2x}\,dx &= \frac{1}{2}\,xe^{2x} - \frac{1}{2}\int e^{2x}\,dx \\ &= \frac{1}{2}\left(x - \frac{1}{2}\right)\,e^{2x}, \end{aligned}$$

and

$$-\int \sin y e^{-\cos y} = -e^{-\cos y}.$$

Thus

$$\frac{1}{2}\left(x - \frac{1}{2}\right)\,e^{2x} = -e^{-\cos y} + c.$$

Thanks to the initial condition, we have

$$-\frac{1}{4} = -1 + c \Leftrightarrow c = \frac{3}{4}.$$

Eventually, the solution of the IVP (3) is given by the relation

$$\frac{1}{2}\left(x - \frac{1}{2}\right)\,e^{2x} = -e^{-\cos y} + \frac{3}{4}.$$

Exercise IV

On the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, the DE in the IVP (4) is equivalent to

$$\tan x\,dx + 2y\,dy = 0,$$

which is obviously equivalent to

$$-\tan x\,dx = 2y\,dy.$$

Integrating this equality, we obtain

$$\ln|\cos x| = y^2 + k,$$

which is equivalent to

$$\ln\cos x = y^2 + k$$

since $\cos x > 0$ for all $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Thus

$$\cos x = e^{y^2+k} \Leftrightarrow e^{-y^2}\cos x = c.$$

Using the initial condition, we have

$$y(0) = 1 \Leftrightarrow e^{-1} = c.$$

That is

$$e^{-y^2} = \frac{1}{e\,\cos x} \Leftrightarrow -y^2 = \ln\left(\frac{1}{e\,\cos x}\right),$$

which is equivalent to

$$y^2 = 1 + \ln\cos x.$$

Now, as $y(0) = 1 > 0$, the function y is nonnegative in an appropriate open interval containing 0. Thus, the solution of the IVP (4) is given by

$$y = \sqrt{1 + \ln\cos x}, \quad \text{on that appropriate interval.}$$

The largest possible interval on which y is defined is

$$(-\cos^{-1}(e^{-1}), \cos^{-1}(e^{-1})).$$

Exercise V

The DE (5) is linear in x, its normal form is

$$\frac{dx}{dy} + \frac{y+2}{y+1}\,x = \frac{2ye^{-y}}{y+1}.$$

Now, we have

$$\begin{aligned}\int \frac{y+2}{y+1}\,dy &= \int \left(1 + \frac{1}{y+1}\right)\,dy \\ &= y + \ln|y+1| + k.\end{aligned}$$

Thus, an integrating factor is given by

$$u(y) = \text{Exp}\left(\int \frac{\text{y}+2}{\text{y}+1}\,\text{dy}\right) = e^y(y+1)$$

Multiplying both sides of the normal form by the integrating factor we obtained above, we get

$$\frac{d}{dy}\left((y+1)\,e^y x\right) = e^y(y+1)\,\frac{2ye^{-y}}{y+1} = 2y.$$

Thus, integrating this equality, we obtain

$$(y+1)e^y x = y^2 + c,$$

and consequently

$$x = \frac{y^2}{y+1}e^{-y} + \frac{c}{y+1}\,e^{-y},$$

is the general solution of the DE (5).

Exercise VI

One may rewrite the DE (6) as

$$\frac{dx}{dy} = \frac{ye^y - 2x}{y} = e^y - \frac{2}{y}x.$$

In particular, we have

$$\frac{dx}{dy} + \frac{2}{y}x = e^y. \tag{10}$$

This clearly a linear (in x) first order differential equation. An integrator factor is given by

$$u(x) = \text{Exp}\left(\int \frac{2}{y}\,dy\right) = y^2.$$

Multiplying both sides of (10) by this integrating factor, we obtain

$$\frac{d}{dy}(y^2 x) = e^y y^2.$$

That is, using integration by parts, we can write

$$\begin{aligned}y^2 x = \int e^y y^2\,dy &= y^2 e^y - 2\int ye^y\,dy \\ &= y^2 e^y - 2\left(ye^y - \int e^y\,dy\right) \\ &= y^2 e^y - 2ye^y + 2e^y + c.\end{aligned}$$

All in all, we obtain that the general solution of the DE (6) is given by

$$x = \left(1 - \frac{2}{y} + \frac{2}{y^2}\right)\,e^y + \frac{c}{y^2}, \quad \forall c \in \mathbb{R}.$$

Exercise VII

The DE (7) is clearly linear. Its normal form is

$$y' + \frac{1}{t}y = t^3\ln t.$$

An integrating factor of this DE is given by

$$\text{Exp}\left(\int \frac{1}{t}\,dt\right) = t.$$

Multiplying the DE written in its normal form above by this integrating factor, we obtain

$$\frac{d}{dt}(ty) = t^4\ln t.$$

This leads to

$$\begin{aligned}ty &= \int t^4\ln t\,dt \\ &= \frac{1}{5}t^5\ln t - \frac{1}{5}\int t^5\,\frac{1}{t}\,dt \\ &= \frac{1}{5}t^5\ln t - \frac{1}{25}t^5 + c.\end{aligned}$$

Eventually, the one-parameter family of solutions of the DE (7) is given by

$$y(t) = \frac{1}{5}t^4\ln t - \frac{1}{25}t^4 + \frac{c}{t}, \quad \forall c \in \mathbb{R}.$$

Exercise VIII

The DE is clearly linear, its normal form is given by

$$\frac{dy}{dx} + \frac{1}{x}y = \frac{2}{x^2(x-2)}. \tag{11}$$

An integrating factor of this DE is given by

$$u(x) = e^{\int \frac{1}{x}\,dx} = x.$$

Multiplying both sides of the DE (11) by x to get

$$(xy)' = \frac{2}{x(x-2)}.$$

That is

$$\begin{aligned} y &= \frac{1}{x}\left(\int \frac{2}{x(x-2)}\,dx\right) \\ &= \frac{1}{x}\left(\int \left(\frac{1}{x-2} - \frac{1}{x}\right)\,ds\right) \\ &= \frac{1}{x}\left(\ln\left|\frac{x-2}{x}\right| + c\right) \end{aligned}$$

Thanks to the initial condition $y(1) = 1$, we have $c = 1$. Thus

$$y = \frac{1}{x}\ln\left|\frac{x-2}{x}\right| + \frac{1}{x}$$

is the solution of the IVP (8).

The interval of validity of the solution is the largest interval I containing 1, which does not contain 0 or 2. Thus, we have clearly

$$I = (0, 2).$$

Exercise IX

On the one side, on $(-\infty, 1)$, the DE in the IVP (9) is equivalent to

$$\frac{dy}{dx} + \frac{2x}{1+x^2}y = \frac{x}{1+x^2}. \qquad (12)$$

. An integrator factor of (12) is given by

$$u(x) = e^{\int \frac{2x}{1+x^2}\,dx} = 1 + x^2.$$

Thus, we have

$$\big((1+x^2)y\big)' = x$$

Thus

$$y = \frac{1}{2}\frac{x^2}{1+x^2} + \frac{c}{1+x^2}.$$

As $y(0) = 0$, we conclude that $c = 0$, and consequently

$$y = \frac{1}{2}\frac{x^2}{1+x^2}, \quad \text{on} \quad (-\infty, 1).$$

On the opposite side, on $(1, +\infty)$, the DE in the IVP (9) is equivalent to

$$\frac{dy}{dx} + \frac{2x}{1+x^2}y = \frac{2x-1}{1+x^2}. \qquad (13)$$

. Proceeding as above, we obtain that the solution on $(1, +\infty)$ is given by

$$y = \frac{x^2 - x + c}{1+x^2}.$$

The solution of the IVP (9) on $\mathbb{R}$ is given by

$$y(x) = \begin{cases} \frac{1}{2}\frac{x^2}{1+x^2} & \text{if} \quad x < 1, \\ \frac{x^2-x+c}{1+x^2} & \text{if} \quad x \geq 1. \end{cases}$$

However as y should be continuous at 1, we get

$$\frac{1}{4} = \frac{1-1+c}{2} = \frac{c}{2},$$

and therefore $c = \frac{1}{2}$. Next, we have to verify that y is differentiable at $x = 1$. For this purpose, we observe that the left derivative of y at 1 is given by

$$\begin{aligned} \lim_{x\to 1^-} \frac{\frac{1}{2}\frac{x^2}{1+x^2} - y(1)}{x-1} &= \lim_{x\to 1^-} \frac{\frac{1}{2}\frac{x^2}{1+x^2} - \frac{1}{4}}{x-1} \\ &= \lim_{x\to 1^-} \frac{1}{4}\frac{1+x}{1+x^2} = \frac{1}{4}, \end{aligned}$$

and the right derivative at 1 is given by

$$\begin{aligned} \lim_{x\to 1^+} \frac{\frac{x^2-x+\frac{1}{2}}{1+x^2} - y(1)}{x-1} &= \lim_{x\to 1^-} \frac{1}{4}\frac{3x^2-4x+1}{(x-1)(1+x^2)} \\ &= \lim_{x\to 1^-} \frac{3}{4}\frac{x-\frac{1}{3}}{1+x^2} = \frac{1}{4}. \end{aligned}$$

We conclude that $y'(1) = \frac{1}{4}$. Thus y is the unique solution of the IVP (9) on $\mathbb{R}$.