King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics

MATH 202 - Exam II - Term 172

Duration: 120 minutes

Name: KEY ID Number:

Section Number: Serial Number:

Class Time: Instructor's Name:

Instructions:

1. Calculators and Mobiles are not allowed.
2. Write legibly.
3. Show all your work. No points for answers without justification.
4. Make sure that you have 8 pages of problems (Total of 12 Problems)
5. **DE means differential equations.**

Question # Number	Points	Maximum Points
1		12
2		4
3		8
4		5
5		6
6		7
7		11
8		14
9		12
10		9
11		6
12		6
Total		100

1. (a) [**8 points**] Verify that the set of functions $\{e^{3x}, x\,e^{3x}\}$ form a fundamental set of solutions of $y'' - 6y' + 9y = 0$ on $(-\infty, \infty)$. [**Do not solve the DE**]

First, we need to prove that $y_1 = e^{3x}$ and $y_2 = xe^{3x}$ are solutions of $y'' - 6y' + 9y = 0$. -- ① Note that, for y_1, we have $y_1 = e^{3x} \Rightarrow y_1' = 3e^{3x} \Rightarrow y_1'' = 9e^{3x}$ and so $LHS = y_1'' - 6y_1' + 9y_1 = 9e^{3x} - 18e^{3x} + 9e^{3x} = 0 = RHS$. 2pts Also, for y_2, we have $y_2 = xe^{3x} \Rightarrow y_2' = e^{3x} + 3xe^{3x} \Rightarrow y_2'' = 6e^{3x} + 9xe^{3x}$ and so $LHS = y_2'' - 6y_2' + 9y_2 = 6e^{3x} + 9xe^{3x} - 6e^{3x} - 18xe^{3x} + 9xe^{3x} = 0 = RHS$. 2pts

$\therefore y_1$ and y_2 are solutions of ①.

Also, we need to prove that y_1 and y_2 are linearly independent.

$$W(y_1, y_2) = \begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & e^{3x} + 3xe^{3x} \end{vmatrix} = e^{6x} + 3xe^{6x} - 3xe^{6x} = e^{6x} \neq 0$$ 1pt for all x. 2pts

So, y_1 and y_2 are linearly independent. That means $\{y_1, y_2\}$ form a fundamental set of solutions of ①. 1pt

(b)[**4 points**] Use part(a) to verify that $y = c_1\,e^{3x} + c_2\,x\,e^{3x} + \frac{x^2}{2}\,e^{3x}$, where c_1 and c_2 are arbitrary constants, is the general solution of the DE $y'' - 6y' + 9y = e^{3x}$ on $(-\infty, \infty)$.

From part (a), we get that $y_c = c_1y_1 + c_2y_2 = c_1e^{3x} + c_2xe^{3x}$. Since $y = y_c + y_p$, then we have $y_p = \frac{x^2}{2}e^{3x}$ and we need to prove that $y_p'' - 6y_p' + 9y_p = e^{3x}$ -- ②. That means, $y_p = \frac{x^2}{2}e^{3x} \Rightarrow y_p' = xe^{3x} + \frac{3}{2}x^2e^{3x} \Rightarrow y_p'' = e^{3x} + 6xe^{3x} + \frac{9}{2}x^2e^{3x}$. So, $LHS = y_p'' - 6y_p' + 9y_p = e^{3x} + \cancel{6xe^{3x}} + \cancel{\frac{9}{2}x^2e^{3x}} - \cancel{6xe^{3x}} - \cancel{9x^2e^{3x}} + \cancel{\frac{9}{2}x^2e^{3x}} = e^{3x} = RHS$. 3pts.

So, y_p is a particular solution of ②.

$\therefore y = c_1y_1 + c_2y_2 + y_p = c_1e^{3x} + c_2xe^{3x} + \frac{x^2}{2}e^{3x}$ is the general solution of ②. 1pt

2. **[4 points]** Given that $y = c_1x^2 + c_2x^4 + 3$ is a two-parameter family of solutions of a differential equation. Determine whether a member of the family satisfies the boundary conditions $y(-1) = 0,\ y(1) = 4$.

$$\left.\begin{array}{l} y(-1) = 0 \Rightarrow c_1 + c_2 + 3 = 0 \Rightarrow c_1 + c_2 = -3 \\ \text{and} \\ y(1) = 0 \Rightarrow c_1 + c_2 + 3 = 4 \Rightarrow c_1 + c_2 = 1 \end{array}\right\}$$ 3 pts

The above system has no solutions and so no member of the family satisfies the boundary conditions. 1 pt

3. **[8 points]** Find a linear second-order differential equation with constant coefficients such that $y_1 = 1$ and $y_2 = e^{-x}$ are solutions of the associated homogeneous equation and $y_p = \frac{1}{2}x^2 - x$ is a particular solution of the nonhomogeneous equation.

Since $y_1 = 1$ and $y_2 = e^{-x}$ are solutions of the homogeneous equation, then their annihilators are, respectively, D and $D+1$. 2 pts 1 pt

So, the homogeneous equation is $D(D+1)y = 0$ and so the associated DE is $y'' + y' = 0$. 1 pt

Since y_p is a particular solution of the nonhomogeneous equation, then $y_p'' + y_p' = f(x)$. 1 pt Since $y_p = \frac{1}{2}x^2 - x \Rightarrow y_p' = x - 1 \Rightarrow y_p'' = 1$

$\Rightarrow y_p'' + y_p' = 1 + x - 1 = x = f(x)$. 2 pts That means the DE is $y'' + y' = x$. 1 pt

4. **[5 points]** Determine whether the set of functions $\{xe^{x+1}, (4x-5)\,e^x, x\,e^x\}$ is linearly independent or linearly dependent on the interval $(-\infty, \infty)$.

Let $y_1 = xe^{x+1}$, $y_2 = (4x-5)e^x$ and $y_3 = xe^x$. Since (2pts)

$y_1 = xe^{x+1} = e\,xe^x = e\,y_3$, then, if we chose

$c_1 = -1$, $c_2 = 0$, and $c_3 = e$, we get (2pts)

$c_1 y_1 + c_2 y_2 + c_3 y_3 = 0$ and so

y_1, y_2 and y_3 are linearly dependent. (1pt.)

5. **[6 points]** Let L be a linear differential operator.
Assume $y_1 = \ln x$, $y_2 = 2\cos x + 4\sin x$, $y_3 = x + 7$ are solutions of the DEs:

$$L(y) = \underbrace{1}_{g_1(x)},\ L(y) = \underbrace{\cos x}_{g_2(x)},\ L(y) = \underbrace{x}_{g_3(x)},\ \text{respectively.}$$

Find a particular solution of the DE:

$$L(y) = \underbrace{\cos^2\left(\frac{x}{2}\right) + 2x + 1}_{g(x)}$$

Since $g(x) = \cos^2\left(\frac{x}{2}\right) + 2x + 1 = \frac{1}{2}\cos x + \frac{1}{2} + 2x + 1$

$= \frac{1}{2}\cos x + 2x + 3/2$

$= \frac{1}{2}g_2(x) + 2g_3(x) + \frac{3}{2}g_1(x)$ -- (3 pts)

Then, by the superposition principle, the solution of

$L(y) = g(x)$ is $y = \frac{1}{2}y_2 + 2y_3 + \frac{3}{2}y_1$ (2pts)

$= \cos x + 2\sin x + 2x + 14 + \frac{3}{2}\ln x$

$= \frac{3}{2}\ln x + \cos x + 2\sin x + 2x + 14.$ (1pt)

6. [**7 points**] Let $y_1 = x^{-1/2}\sin x$ be a solution of $x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0$. Use the reduction of order formula to find a ~~second~~ solution of this differential equation.

1pt

In the DE, $P(x) = \frac{x}{x^2} = \frac{1}{x}$. By the reduction of order formula, we get $y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{y_1^2}dx$ 2pts

$$= x^{-\frac{1}{2}}\sin x \int \frac{e^{-\int \frac{1}{x}dx}}{x^{-1}\sin^2 x}dx$$

$$= x^{-\frac{1}{2}}\sin x \int \frac{x^{-1}}{x^{-1}\sin^2 x}dx = x^{-\frac{1}{2}}\sin x \int \csc^2 x\,dx$$

$$= x^{-\frac{1}{2}}\sin x(-\cot x) = -x^{-1/2}\cos x$$ 4pts

7. [**11 points**] Given that $y = \sin x$ is a solution of $y^{(4)} + 2y''' + 11y'' + 2y' + 10y = 0$. Find the general solution of the DE.

Since $y_1 = \sin x$ is a solution, $y_2 = \cos x$ is also a solution and so i and $-i$ are zeros of the auxiliary equation of the DE:

$$m^4 + 2m^3 + 11m^2 + 2m + 10 = 0.$$ 2pts

By the synthetic division, we get

$$\begin{array}{r|rrrrr} i & 1 & 2 & 11 & 2 & 10 \\ & & i & -1+2i & -2+10i & -10 \\ \hline -i & 1 & 2+i & 10+2i & 10i & 0 \\ & & -i & -2i & -10i & \\ \hline & 1 & 2 & 10 & 0 & \end{array}$$

4pts

The zeros of the equation $m^2 + 2m + 10 = 0$ is

$$m = \frac{-2 \pm \sqrt{4-40}}{2} = -1 \pm 3i$$ 2pts

2pts

So, $y_3 = e^{-x}\sin 3x$ and $y_4 = e^{-x}\cos 3x$ and so the general solution is:

$$y = c_1y_1 + c_2y_2 + c_3y_3 + c_4y_4.$$ 1pt

8. **[14 points]** Solve the following differential equation by using undetermined coefficients-annihilator approach

$$y'' + y = \cos x.$$

First, we solve $y''+y=0$. We get the auxiliary equation

$m^2+1=0 \Rightarrow m=\pm i$ and so $y_c = c_1 \cos x + c_2 \sin x$. (2pts)

(1pt)

The annihilator of $\cos x$ is D^2+1 and so, if

we annihilates the DE $y''+y=\cos x$, we get

$(D^2+1)(D^2+1)y = (D^2+1)(\cos x) = 0$ (1pt)

So, the solution of the new equation is

$\tilde{y} = c_1 \cos x + c_2 \sin x + c_3 x \cos x + c_4 x \sin x$ (2pts)

By deleting y_c, we get $y_p = Ax\cos x + Bx\sin x$ (1pt)

$\Rightarrow y_p' = A\cos x - Ax\sin x + B\sin x + Bx\cos x$

$\Rightarrow y_p'' = -A\sin x - A\sin x - Ax\cos x + B\cos x + B\cos x - Bx\sin x$

$= -2A\sin x + 2B\cos x - Ax\cos x - Bx\sin x$

So, $y_p'' + y_p = -2A\sin x + 2B\cos x - Ax\cos x - Bx\sin x + Ax\cos x + Bx\sin x$

$= -2A\sin x + 2B\cos x = \cos x + 0\sin x$ (3pts)

$\Rightarrow -2A = 0$ and $2B = 1 \Rightarrow A = 0$ and $B = \frac{1}{2}$ (2pts)

$\Rightarrow y_p = \frac{1}{2} x \sin x$ (1pt)

So, the general solution is $y = y_c + y_p = c_1 \cos x + c_2 \sin x + \frac{1}{2} x \sin x$

(1pt)

9. [**12 points**] Find the general solution of the differential equation

$$y'' + 8y' + 16y = \frac{e^{-4x}}{x^2}, \quad x > 0.$$

First, we solve $y''+8y'+16y=0$. We get the auxiliary equation $m^2+8m+16=(m+4)^2=0 \Rightarrow m=-4,-4$ and so $y_c = c_1 e^{-4x} + c_2 x e^{-4x}$. (2pts)

By using variation of parameters, we have

$y_1 = e^{-4x}$, $y_2 = xe^{-4x}$ and $f(x) = \dfrac{e^{-4x}}{x^2}$ (1pt)

So, $W = W(y_1, y_2) = \begin{vmatrix} e^{-4x} & xe^{-4x} \\ -4e^{-4x} & e^{-4x} - 4xe^{-4x} \end{vmatrix} = e^{-8x} \neq 0$ (1pt)

Also, $W_1 = \begin{vmatrix} 0 & xe^{-4x} \\ \dfrac{e^{-4x}}{x^2} & e^{-4x} - 4xe^{-4x} \end{vmatrix} = \dfrac{-e^{-8x}}{x}$ (1pt) and $W_2 = \begin{vmatrix} e^{-4x} & 0 \\ -4e^{-4x} & \dfrac{e^{-4x}}{x^2} \end{vmatrix} = \dfrac{e^{-8x}}{x^2}$. (1pt)

So, $u_1' = \dfrac{W_1}{W} = \dfrac{-e^{-8x}}{xe^{-8x}} = -\dfrac{1}{x} \Rightarrow u_1 = -\ln x$ (2pts)

and $u_2' = \dfrac{W_2}{W} = \dfrac{e^{-8x}}{x^2 e^{-8x}} = \dfrac{1}{x^2} \Rightarrow u_2 = -\dfrac{1}{x}$ (2pts)

So, $y_p = u_1 y_1 + u_2 y_2 = (-\ln x)e^{-4x} - \dfrac{1}{x} xe^{-4x} = -e^{-4x}\ln x - e^{-4x}$ (1pt)

$\therefore$ The general solution is

$$y = y_c + y_p = c_1 e^{-4x} + c_2 xe^{-4x} - e^{-4x}\ln x - e^{-4x}$$ (1pt)

$$= c_1 e^{-4x} + c_2 xe^{-4x} - e^{-4x}\ln x$$

10. [**9 points**] Solve the DE:

$$x^3 y''' + 2x y' - 2y = 0, \quad x > 0.$$

This is a Cauchy-Euler equation. If we let $y = x^m$, (1pt)

then we get $y' = m x^{m-1} \Rightarrow y'' = m(m-1) x^{m-2} \Rightarrow y''' = m(m-1)(m-2) x^{m-3}$.

So, the DE becomes

$$m(m-1)(m-2) x^m + 2m x^m - 2x^m = 0$$

So, the auxiliary equation is $m(m^2 - 3m + 2) + 2m - 2 = 0$ (3pts)

$\Rightarrow m^3 - 3m^2 + 4m - 2 = 0$. It is clear $m = 1$ is a solution

and so

1	1	-3	4	-2
		1	-2	2
	1	-2	2	0

(1pt)

The zeroes of $m^2 - 2m + 2 = 0$ are $m = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$ (2pts)

So, the zeroes of the auxiliary equation are $1, 1 \pm i$

So, the general solution of the DE is

$$y = c_1 x + c_2 x \cos(\ln x) + c_3 x \sin(\ln x)$$ (2pts)

11. **[6 points]** Use a suitable substitution to transform the DE $xy'' + \frac{y}{x} = \frac{\tan^{-1}(\ln x)}{x}$, $x > 0$, into a linear DE with constant coefficients. **(Do not solve the new equation)**.

If we multiply by x, we get $x^2y'' + y = \tan^{-1}(\ln x)$ 1pt

which is a Cauchy-Euler equation. 1pt Now, if we use

$x = e^t$, then $t = \ln x$ and so $x\frac{dy}{dx} = \frac{dy}{dt}$ and $x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$ 2pts

So, the DE becomes

$\frac{d^2y}{dt^2} - \frac{dy}{dt} + y = \tan^{-1} t$ 2pts which is a

linear DE with constant coefficients.

12. **[6 points]** Find a differential operator that annihilates

$$5 + 6x^3 - e^x + 5x\,e^{-x} - x^2\,e^{-x} + e^x \sin 4x + e^x\,x^2 \cos 4x$$

The annihilator of $5+6x^3$ is D^4 1pt

" " " $-e^x$ is $D-1$ 1pt

" " " $5xe^{-x} - x^2e^{-x}$ is $(D+1)^3$ 1pt

" " " $e^x\sin 4x + e^x x^2 \cos 4x$ is $(D^2-2D+17)^3$ 2pts

So, $L = D^4(D-1)(D+1)^3(D^2-2D+17)^3$ 1pt will annihilate

$5+6x^3 - e^x + 5xe^{-x} - x^2e^{-x} + e^x\sin 4x + e^x x^2 \cos 4x$.