King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics

MATH 202 - Exam I - Term 172

Duration: 120 minutes

Name: KEY ID Number:

Section Number: Serial Number:

Class Time: Instructor's Name:

Instructions:

1. Calculators and Mobiles are not allowed.
2. Write legibly.
3. Show all your work. No points for answers without justification.
4. Make sure that you have 8 pages of problems (Total of 10 Problems)
5. **DE means differential equations.**

Question # Number	Points	Maximum Points
1		5
2		7
3		17
4		10
5		11
6		10
7		6
8		7
9		13
10		14
Total		100

1. **[5 points]** Find the solution of the initial-value problem

$$y'' + 4y = 0$$

$$y(0) = 0,\ y'(0) = 1$$

given that the general solution of the given differential equation is

$$y(x) = c_1 \sin(2x) + c_2 \cos(2x).$$

$\Rightarrow y'(x) = 2c_1 \cos(2x) - 2c_2 \sin(2x)$ 1 pt.

$y(0) = 0 \Rightarrow c_2(1) = 0 \Rightarrow c_2 = 0$

$y'(0) = 1 \Rightarrow 2c_1 = 1 \Rightarrow c_1 = \frac{1}{2}$ 3 pts

$\Rightarrow y(x) = \frac{1}{2}\sin(2x)$ 1 pt.

2. **[7 points]** Use the existence and uniqueness theorem to find the values of β so that the initial-value problem

$$\frac{dy}{dx} = \frac{\ln(e^y - e^{2x})}{\sqrt{4 - x^2}}, \quad y(1) = \beta,$$

has a unique solution.

$f(x,y) = \dfrac{\ln(e^y - e^{2x})}{\sqrt{4-x^2}}$ 1 pt.

$\Rightarrow \dfrac{\partial f}{\partial y} = \dfrac{e^y}{(e^y - e^{2x})\sqrt{4-x^2}}$ 1 pt.

f and $\frac{\partial f}{\partial y}$ are continuous if $4 - x^2 > 0$ and $e^y - e^{2x} > 0$ 2 pts.

That means $-2 < x < 2$ and $e^y > e^{2x}$ which implies $y > 2x$. 2 pts.

From $y(1) = \beta$, we get $x = 1$ and $y = \beta$

So, since $-2 < 1 < 2$, we need $\beta > 2(1) = 2$. 1 pt.

i.e., $\beta \in (2, \infty)$.

3. Given the DE $(y^2 - 4)\,dx + 4x^2\,dy = 0$.

(a) [**7 points**] Find the general solution of the above DE and rewrite it in an **explicit form**.

$$(y^2-4)\,dx = -4x^2 dy \Rightarrow \frac{dx}{-4x^2} = \frac{dy}{y^2-4} = \frac{dy}{(y-2)(y+2)} = \frac{1}{4}\left(\frac{1}{y-2} - \frac{1}{y+2}\right)dy$$ 2pts. 2pts.

Integrate to get $\frac{1}{x} + c = (\ln|y-2| - \ln|y+2|) = \ln\left|\frac{y-2}{y+2}\right|$

$$e^{\frac{1}{x}+c} = \left|\frac{y-2}{y+2}\right| \Rightarrow \frac{y-2}{y+2} = ce^{\frac{1}{x}}$$ 1pt. $\Rightarrow cye^{\frac{1}{x}} + 2ce^{\frac{1}{x}} = y-2$

$$\Rightarrow y(ce^{\frac{1}{x}} - 1) = -2 - 2ce^{\frac{1}{x}} \Rightarrow y = \frac{2(1+ce^{\frac{1}{x}})}{1-ce^{\frac{1}{x}}}$$ 2pts.

(b) [**4 points**] If $y = k$ is a constant solution of the above DE, then find the possible **value(s)** of k.

2pts

$$y = k \Rightarrow \frac{dy}{dx} = \frac{y^2-4}{-4x^2} = 0 \Rightarrow y^2 - 4 = 0$$ 1pt.

$$\Rightarrow y = \pm 2$$ 1pt.

(c) [**6 points**] Using parts (a) and (b), find all singular solutions.

If $y = 2 = \frac{2(1+ce^{\frac{1}{x}})}{1-ce^{\frac{1}{x}}}$ 1pt., then we get $2 - 2ce^{\frac{1}{x}} = 2 + 2ce^{\frac{1}{x}}$

$\Rightarrow 4ce^{\frac{1}{x}} = 0 \Rightarrow c = 0$ 1pt. $\Rightarrow y = 2$ is not a singular soln. 1pt.

If $y = -2 = \frac{2(1+ce^{\frac{1}{x}})}{1-ce^{\frac{1}{x}}}$ 1pt., then we get $-2 + 2ce^{\frac{1}{x}} = 2 + 2ce^{\frac{1}{x}}$

$\Rightarrow -2 = 2$ which is impossible 1pt.

$\Rightarrow y = -2$ is a singular solution

1pt.

4. **[10 points]** Find the solution of initial-value problem

$$\begin{cases} y' + 2xy = x^3, \\ y(0) = 0. \end{cases}$$

$y' + 2xy = x^3$ is a linear DE (1 pt.)

$$\text{I.F.} = e^{\int 2x\,dx} = e^{x^2} \quad \text{(1 pt.)}$$

$$\Rightarrow e^{x^2} y' + 2x e^{x^2} y = x^3 e^{x^2}$$

$$\Rightarrow \frac{d}{dx}\left[e^{x^2} y\right] = x^3 e^{x^2} \quad \text{(2 pts.)}$$

Integrate $\Rightarrow$ $e^{x^2} y = \int x^3 e^{x^2} dx = \int \frac{x^2}{2}(2x e^{x^2}) dx$

$u = \frac{x^2}{2}$ $\quad dv = 2xe^{x^2} dx$

$du = x\,dx$ $\quad V = e^{x^2}$

$$\Rightarrow e^{x^2} y = \frac{x^2}{2} e^{x^2} - \int x e^{x^2} dx$$

$$= \frac{x^2}{2} e^{x^2} - \frac{1}{2} e^{x^2} + C \quad \text{(3 pts.)}$$

$$\Rightarrow y = \frac{x^2}{2} - \frac{1}{2} + c e^{-x^2} \quad \text{(1 pt.)}$$

$y(0) = 0 \Rightarrow -\frac{1}{2} + C = 0 \Rightarrow C = \frac{1}{2}$ (1 pt.) $\Rightarrow y = \frac{x^2}{2} - \frac{1}{2} + \frac{1}{2} e^{-x^2}$ (1 pt.)

5. For the differential equation

$$(y^3 + kxy^4 - 2x)\,dx + (3xy^2 + 20x^2y^3)\,dy = 0,$$

(a) [**3 points**] Find the value of k so that the differential equation is exact.

$M(x,y) = y^3 + kxy^4 - 2x$ & $N(x,y) = 3xy^2 + 20x^2y^3$

The DE is exact means $\frac{\partial M}{\partial y} = 3y^2 + 4kxy^3 = \frac{\partial N}{\partial x} = 3y^2 + 40xy^3$ 1pt. 1pt.

$\Rightarrow 4k = 40 \Rightarrow k = 10$ 1pt.

(b) [**8 points**] Use the value of k from part (a) to find the general solution of the above differential equation.

The DE is $(y^3 + 10xy^4 - 2x)\,dx + (3xy^2 + 20x^2y^3)\,dy = 0$

We start with

1pt.

$\frac{\partial f}{\partial y} = 3xy^2 + 20x^2y^3 \Rightarrow f(x,y) = xy^3 + 5x^2y^4 + h(x)$ 2pts.

1pt.

$\Rightarrow \frac{\partial f}{\partial x} = y^3 + 10xy^4 + h'(x) = M(x,y) = y^3 + 10xy^4 - 2x$

$\Rightarrow h'(x) = -2x \Rightarrow h(x) = -x^2$ 2pts.

$\Rightarrow f(x,y) = xy^3 + 5x^2y^4 - x^2$ 1pt.

Soln $xy^3 + 5x^2y^4 - x^2 = C$ 1pt.

6. **[10 points]** Consider the DE

$$(xy)\,dx + (2x^2 + 3y^2 - 20)\,dy = 0, \qquad y > 0.$$

(a) Find an integrating factor that transforms the above DE to an exact equation. **(Do not solve the new equation).**

$M(x,y) = xy$ & $N(x,y) = 2x^2 + 3y^2 - 20$

$\frac{\partial M}{\partial y} = x \neq \frac{\partial N}{\partial x} = 4x$ 2 pts.

However, $\frac{N_x - M_y}{M} = \frac{3x}{xy} = \frac{3}{y}$ (function of y only) 2 pts.

$= f(y)$

$\Rightarrow$ I.F. $= e^{\int f(y)dy}$ 1 pt. $= e^{\int \frac{3}{y} dy} = e^{3\ln|y|} = y^3$ 2 pts.

The new equation is $xy^4 dx + (2x^2y^3 + 3y^5 - 20y^3)dy = 0$ 1 pt.

(b) Show that the new equation is an exact equation.

$\tilde{M}(x,y) = xy^4$ & $\tilde{N}(x,y) = 2x^2y^3 + 3y^5 - 20y^3$

Since $\frac{\partial \tilde{M}}{\partial y} = 4xy^3 = \frac{\partial \tilde{N}}{\partial x} = 4xy^3$, then the new 2 pts.

equation is exact.

7. **[6 points]** Use a suitable substitution to transform the DE

$$\frac{1}{z}\frac{dz}{dx} + \frac{1}{x}\ln z = 3x$$

to a linear DE **(Do not solve the new equation).**

Let $y = \ln z$ 2 pts. $\Rightarrow \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}$

$= \frac{1}{z}\frac{dz}{dx}$ 2 pts.

$\Rightarrow$ The DE becomes $\frac{dy}{dx} + \frac{1}{x}y = 3x$ linear 2 pts.

8. [**7 points**] Use a suitable substitution to transform the DE

$$\frac{dy}{dx} = \frac{y}{x}(\ln y - \ln x + 1).$$

to a seperable equation (**Do not solve the new equation**).

$(\ln y - \ln x + 1)\,dx - \frac{x}{y}\,dy = 0$ 1 pt.

Let $M(x,y) = \ln y - \ln x + 1$ & $N(x,y) = -\frac{x}{y}$

Note $M(tx,ty) = \ln ty - \ln tx + 1 = \ln \frac{y}{x} + 1 = \ln y - \ln x + 1 = M(x,y)$

$N(tx,ty) = -\frac{tx}{ty} = \frac{-x}{y} = N(x,y)$

The DE is homogeneous 2 pts.

Let $y = vx \Rightarrow dy = v\,dx + x\,dv$ 2 pts.

$\Rightarrow$ The DE becomes $(\ln(vx) - \ln x + 1)\,dx - \frac{1}{v}(v\,dx + x\,dv) = 0$ 1 pt.

$\Rightarrow (\ln v)\,dx + \cancel{dx} - \cancel{dx} - \frac{x}{v}\,dv = 0$

$\Rightarrow dx(\ln v) = \frac{x}{v}\,dv$

$\Rightarrow \frac{dx}{x} = \frac{1}{v\ln v}\,dv$ seperable 1 pt.

9. [**13 points**] Solve the initial value problem

$$t^2\frac{dx}{dt} = x\left(\frac{1}{2}t - 3x^2\right), \qquad x(1) = 1, \qquad t > 0$$

$$\Rightarrow \frac{dx}{dt} - \frac{1}{2t}x = \frac{-3}{t^2}x^3$$ 1 pt.

Bernoulli with $n = 3$

Let $u = x^{-2} \Rightarrow u^{-1/2} = x \Rightarrow \frac{dx}{dt} = \frac{-1}{2}u^{-3/2}\frac{du}{dt}$ 3 pts.

The DE becomes

$$-\frac{1}{2}u^{-3/2}\frac{du}{dt} - \frac{1}{2t}u^{-1/2} = \frac{-3}{t^2}u^{-3/2}$$ 1 pt.

Multiply by $-2u^{+3/2}$ to get $\frac{du}{dt} + \frac{1}{t}u = \frac{6}{t^2}$ linear in u 1 pt.

$\text{I.F.} = e^{\int \frac{1}{t}dt} = t$ 1 pt. $\Rightarrow t\frac{du}{dt} + u = \frac{6}{t}$

$\Rightarrow \frac{d}{dt}(tu) = \frac{6}{t} \Rightarrow tu = 6\ln t + c \Rightarrow u = \frac{6}{t}\ln t + \frac{c}{t}$ 3 pts.

If $t = 1$, then $x = 1 \Rightarrow u = 1 \Rightarrow 1 = 0 + \frac{c}{1} \Rightarrow c = 1$ 2 pts

$\Rightarrow u = \frac{6}{t}\ln t + \frac{1}{t} \Rightarrow x^{-2} = \frac{6}{t}\ln t + \frac{1}{t}$ 1 pt

10. **[14 points]** Take a can of soda from the refrigerator (temperature of the can of soda inside the refrigerator being the same as of the refrigerator) and let it warm for an hour. Record the temperature twice i.e. after every thirty minutes. Suppose that the recordings turn out to be 45° F and 55° F, respectively. Assuming the room temperature as 70° F, what was the temperature inside the refrigerator when the can was taken out?

Let T be the temperature and we have

$T_m = 70$ & $T(\frac{1}{2}) = 45$ & $T(1) = 55$ 3pts

we would like to find $T_0 = T(0)$.

Also, we have the DE $\frac{dT}{dt} = k(T - 70)$

$\Rightarrow \frac{dT}{T-70} = k\,dt \Rightarrow \ln|T-70| = kt + C$

$\Rightarrow T = 70 + Ce^{kt}$ 3pts.

$T(\frac{1}{2}) = 45 \Rightarrow 45 = 70 + Ce^{\frac{1}{2}k} \Rightarrow -25 = Ce^{\frac{1}{2}k}$ — (1)

$T(1) = 55 \Rightarrow 55 = 70 + Ce^{k} \Rightarrow -15 = Ce^{k}$ — (2)

3pts

From (1) and (2), we get $\frac{25}{15} = e^{-\frac{1}{2}k} \Rightarrow \frac{5}{3} = e^{-\frac{1}{2}k}$ 2pts

$\Rightarrow k = -2\ln 5/3 = 2\ln 3/5$

Substitute in (1), to get $-25 = Ce^{\ln(\frac{3}{5})} = C(\frac{3}{5})$

$\Rightarrow C = \frac{-125}{3}$ 2pts

So, $T(0) = 70 - \frac{125}{3}e^{(0)} = \frac{210 - 125}{3} = \frac{85}{3}$ 1pt