(1)

Math201.01, Quizzes #3 & 4, Term 172

Name:

ID #: Solutions **Serial #:**

1. [7 points] Find the local maximum and minimum values and saddle points of $f(x,y) = y^3 + 3x^2y - 6x^2 - 6y^2 + 2$.

• $f_x(x,y) = 6xy - 12x$; $f_y(x,y) = 3y^2 + 3x^2 - 12y$

$$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \Longrightarrow \begin{cases} 6xy - 12x = 0 \\ 3y^2 + 3x^2 - 12y = 0 \end{cases} \Longrightarrow \begin{cases} xy - 2x = 0 & \text{— (1)} \\ y^2 + x^2 - 4y = 0 & \text{— (2)} \end{cases}$$ (1)

(1) $\Longrightarrow x(y-2) = 0 \Longrightarrow \underline{x = 0 \text{ or } y = 2}$

• $x = 0 \xrightarrow{(2)} y^2 - 4y = 0 \Longrightarrow y(y-4) = 0 \Longrightarrow y = 0$ or $y = 4$

Ct. pts $\boxed{(x,y) = (0,0), (0,4)}$ $\in$ Domain of f (1)

• $y = 2 \xrightarrow{(2)} 4 + x^2 - 8 = 0 \Longrightarrow x^2 = 4 \Longrightarrow x = \pm 2$

Ct pts $\boxed{(x,y) = (-2,2), (2,2)}$ $\in$ domain of f (1)

• f_x & f_y exist at all pts (x,y).

• Test: $f_{xx}(x,y) = 6y - 12$; $f_{yy}(x,y) = 6y - 12$; $f_{xy}(x,y) = 6x$ (1)

$D(x,y) = f_{xx} f_{yy} - (f_{xy})^2 = (6y-12)^2 - (6x)^2 \stackrel{\text{or}}{=} 36\left[(y-2)^2 - x^2\right]$

(1) • $D(0,0) = 36(4) = 144 > 0$ & $f_{xx}(0,0) = -12 < 0 \Rightarrow$ f has a local max at $(0,0)$

(1) • $D(0,4) = 36(4) = 144 > 0$ & $f_{xx}(0,4) = 24 - 12 = 12 > 0 \Rightarrow$ ______ min at $(0,4)$

(0.5) • $D(-2,2) = 36(-4) = -144 < 0 \Rightarrow$ f has a saddle pt at $(-2,2)$.

(0.5) • $D(2,2) = 36(-4) = -144 < 0 \Rightarrow$ ______ .

→ The local max value of f is $f(0,0) = 2$

→ ______ min ______ is $f(0,4) = 4^3 - 6(4)^2 + 2 = 64 - 96 + 2 = -30$

2

2. [5 points] Find the extreme values of $\cancel{f(x,y)}$ $f(x,y,z)$ $= 2x + 2y + z$ on the sphere $x^2 + y^2 + z^2 = 9$.

We use Lagrange Multipliers.

Take $g(x,y,z) = x^2+y^2+z^2-9$. We solve the system

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ f_z = \lambda g_z \\ g = 0 \end{cases} \Longrightarrow \begin{cases} 2 = \lambda(2x) & \text{——— (1)} \\ 2 = \lambda(2y) & \text{——— (2)} \\ 1 = \lambda(2z) & \text{——— (3)} \\ x^2+y^2+z^2-9=0 & \text{——— (4)} \end{cases}$$

1.5

1

$(1),(2),(3) \Longrightarrow x = \frac{1}{\lambda},\ y = \frac{1}{\lambda},\ z = \frac{1}{2\lambda}$ ——— (5) $,\ \lambda \neq 0$
$[\text{If } \lambda = 0 \overset{(1)}{\Longrightarrow} 2 = 0,$ impossible $]$

$$\overset{(4)}{\Longrightarrow} \frac{1}{\lambda^2} + \frac{1}{\lambda^2} + \frac{1}{4\lambda^2} - 9 = 0$$

$$\overset{4\lambda^2}{\Longrightarrow} 4 + 4 + 1 - 36\lambda^2 = 0$$

$$\Longrightarrow 36\lambda^2 = 9 \Longrightarrow \lambda^2 = \frac{1}{4} \Longrightarrow \lambda = \pm\frac{1}{2}$$ 0.5

- $\lambda = -\frac{1}{2} \overset{(5)}{\Longrightarrow} (x,y,z) = (-2,-2,-1)$ 0.5
- $\lambda = \frac{1}{2} \overset{(5)}{\Longrightarrow} (x,y,z) = (2,2,1)$ 0.5

- $f(-2,-2,-1) = -4-4-1 = -9$, the min. value of f 0.5

  $f(2,2,1) = 4+4+1 = 9$, the max. value of f 0.5

3

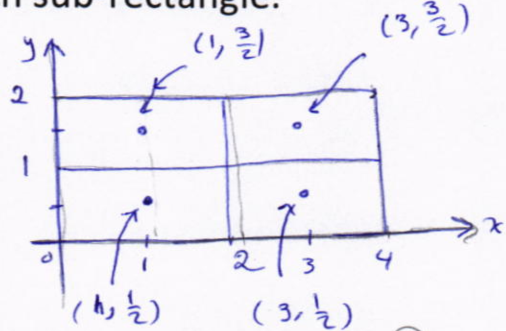
3. [4 points] Let $R = [0,4] \times [0,2]$. Estimate the double integral $\iint_R \ln(x+2y)\,dA$ using double Riemann sums with $m = 2$ *and* $n = 2$ and taking as sample points the center of each sub-rectangle.

$\Delta x = \frac{4-0}{m} = \frac{4}{2} = 2$

$\Delta y = \frac{2-0}{n} = \frac{2}{2} = 1$

$\Delta A = \Delta x \, \Delta y = (2)(1) = 2$ 0.5

$f(x,y) = \ln(x+2y)$



$$\iint_R f(x,y)\,dA \approx f(1,\tfrac{1}{2})\Delta A + f(1,\tfrac{3}{2})\Delta A + f(3,\tfrac{1}{2})\Delta A + f(3,\tfrac{3}{2})\Delta A$$ 2

$= \ln 2 \cdot 2 + \ln 4 \cdot 2 + \ln 4 \cdot 2 + \ln 6 \cdot 2$

$= 2(\ln 2 + \ln 4 + \ln 4 + \ln 6)$; $\ln 4 = 2\ln 2$ & $\ln 6 = \ln 2 + \ln 3$

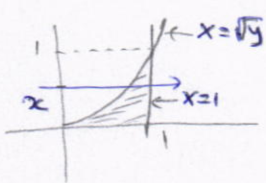
$= 2(\ln 2 + 2\ln 2 + 2\ln 2 + \ln 2 + \ln 3)$

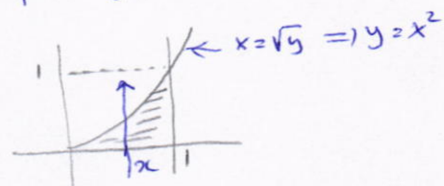
$= 2(6\ln 2 + \ln 3)$ 0.5

4. [4 points] Evaluate $\int_0^1 \int_{\sqrt{y}}^1 \sqrt{x^3+1}\;dx\,dy$.

. Not easy to integrate. So Reverse the order of integration

. $R: \sqrt{y} \le x \le 1,\ 0 \le y \le 1$ (Type II)





$R: 0 \le y \le x^2,\ 0 \le x \le 1$ (Type I)

$$\int_0^1 \int_{\sqrt{y}}^1 \sqrt{x^3+1}\;dx\,dy = \int_0^1 \int_0^{x^2} \sqrt{x^3+1}\;dy\,dx$$ 2

$$= \int_0^1 \sqrt{x^3+1}\cdot y\Big]_{y=0}^{y=x^2} dx$$ 0.5

$$= \int_0^1 \sqrt{x^3+1}\cdot x^2\,dx$$ 0.5, let $u = x^3+1 \Rightarrow du = 3x^2\,dx$

$x=0 \Rightarrow u=1$

$x=1 \Rightarrow u=2$

$$= \frac{1}{3}\int_1^2 \sqrt{u}\,du$$

$$= \frac{1}{3}\cdot\frac{2}{3}\,u^{3/2}\Big]_1^2$$ 0.5

$$= \frac{2}{9}\left(\sqrt{8}-1\right)$$ 0.5

4

Math201.02, Quizzes #3 & 4, Term 172

Name:

ID #: **Serial #:**

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1. [6 points] Find the local maximum and minimum values and saddle points of $f(x,y) = 2x^2 + y^4 + 4xy - 1$.

- $f_x(x,y) = 4x + 4y$; $f_y(x,y) = 4y^3 + 4x$

$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \Rightarrow \begin{cases} 4x + 4y = 0 \\ 4y^3 + 4x = 0 \end{cases} \Rightarrow \begin{cases} x + y = 0 & \text{—— (1)} \\ y^3 + x = 0 & \text{—— (2)} \end{cases}$ 1

$(1) \Rightarrow \boxed{y = -x}$ (3)

$\xRightarrow{(2)} (-x)^3 + x = 0 \Rightarrow -x^3 + x = 0 \Rightarrow -x(x^2-1) = 0 \Rightarrow -x(x-1)(x+1) = 0$

$\Rightarrow x = 0, -1, 1$ ct pts 1.5

- $x = 0 \xRightarrow{(3)} y = 0 \Rightarrow \boxed{(x,y) = (0,0)}$
- $x = -1 \xRightarrow{(3)} y = 1 \Rightarrow \boxed{(x,y) = (-1,1)}$ $\in$ domain of f
- $x = 1 \xRightarrow{(3)} y = -1 \Rightarrow \boxed{(x,y) = (1,-1)}$

- f_x & f_y exist at all pts (x,y).

<u>Test</u>: $f_{xx}(x,y) = 4$; $f_{yy}(x,y) = 12y^2$; $f_{xy}(x,y) = 4$ 1

$D(x,y) = f_{xx}f_{yy} - (f_{xy})^2 = (4)(12y^2) - (4)^2 = 48y^2 - 16 \overset{\text{or}}{=} 16(3y^2-1)$

0.5
- $D(0,0) = -16 < 0 \Rightarrow f$ has a saddle pt at $(0,0)$
- $D(-1,1) = 16(3-1) = 32 > 0$ & $f_{xx}(-1,1) = 4 > 0 \Rightarrow f$ has a local min at $(-1,1)$
- $D(1,-1) = 16(3-1) = 32 > 0$ & $f_{xx}(1,-1) = 4 > 0 \Rightarrow$ ______ $(1,-1)$

- the local min values of f ~~are~~ is $f(-1,1) = -2 = f(1,-1)$

2. [6 points] Use Lagrange Multipliers to find the extreme values of $f(x,y) = y^2 - x^2 + x$ on the ellipse $x^2 + 4y^2 = 4$.

we take $g(x,y) = x^2+4y^2-4$. we solve the system

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ g=0 \end{cases} \Longrightarrow \begin{cases} -2x+1 = \lambda(2x) \\ 2y = \lambda(8y) \\ x^2+4y^2=4 \end{cases} \Longrightarrow \begin{cases} -2x+1 = 2\lambda x & \text{——— (1)} \\ y = 4\lambda y & \text{——— (2)} \\ x^2+4y^2 = 4 & \text{——— (3)} \end{cases}$$ 1

$(2) \Longrightarrow y - 4\lambda y = 0 \Longrightarrow y(1-4\lambda) = 0 \Longrightarrow \boxed{y=0}$ or $\boxed{\lambda = \frac{1}{4}}$ 1

- $y = 0 \overset{(3)}{\Longrightarrow} x^2 + 0 = 4 \Longrightarrow x = \pm 2$ 1

  we get the points $\boxed{(x,y) = (-2,0),\ (2,0)}$

- $\lambda = \frac{1}{4} \overset{(1)}{\Longrightarrow} -2x+1 = \frac{1}{2}x \Longrightarrow -4x+2 = x \Longrightarrow 5x = 2 \Longrightarrow x = \frac{2}{5}$

  $\overset{(3)}{\Longrightarrow} \frac{4}{25} + 4y^2 = 4 \Longrightarrow \frac{1}{25} + y^2 = 1 \Longrightarrow y^2 = \frac{24}{25} \Longrightarrow y = \pm\frac{2\sqrt{6}}{5}$

  we get the points $(x,y) = \left(\frac{2}{5}, \frac{-2\sqrt{6}}{5}\right), \left(\frac{2}{5}, \frac{2\sqrt{6}}{5}\right)$ 1

- $f(-2,0) = 0 - 4 - 2 = -6$

  $f(2,0) = 0 - 4 + 2 = -2$

$f\left(\frac{2}{5}, \frac{-2\sqrt{6}}{5}\right) = \frac{24}{25} - \frac{4}{25} + \frac{2}{5} = \frac{20}{25} + \frac{10}{25} = \frac{30}{25} = \frac{6}{5}$

$f\left(\frac{2}{5}, \frac{2\sqrt{6}}{5}\right) = \frac{30}{25} = \frac{6}{5}$

- The max value of f is $\frac{6}{5}$ & it occurs at the pts $\left(\frac{2}{5}, \pm\frac{2\sqrt{6}}{5}\right)$ 1

  The min value of f is -6 & ——————— pt $(-2,0)$ 1

3. [4 points] Find the average value of $f(x,y) = \frac{1}{\sqrt{1+x+y}}$ over the region $R = [0,2] \times [0,1]$.

• R is a rectangular region. So area$(R) = (2-0)(1-0) = 2$ 0.5

• $f_{ave} = \frac{1}{\text{area}(R)} \iint_R f(x,y)\, dA$ 0.5

$= \frac{1}{2} \int_0^2 \int_0^1 (1+x+y)^{-1/2}\, dy\, dx$ 1

$= \frac{1}{2} \int_0^2 2(1+x+y)^{1/2} \Big]_{y=0}^{y=1}$ 0.5

$= \frac{1}{2} \int_0^2 2 \cdot \left[(2+x)^{1/2} - (1+x)^{1/2}\right] dx$ 0.5

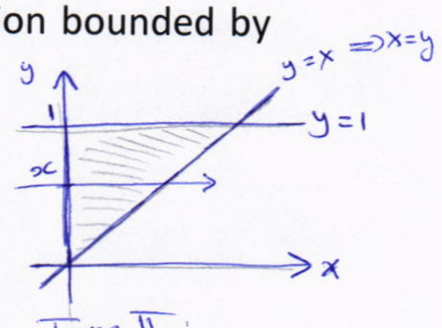
$= \int_0^2 \left[(2+x)^{1/2} - (1+x)^{1/2}\right] dx$

$= \frac{2}{3}(2+x)^{3/2} - \frac{2}{3}(1+x)^{3/2} \Big]_0^2$ 0.5

$= \frac{2}{3}\left[(4^{3/2} - 3^{3/2}) - (2^{3/2} - 1)\right]$

$= \frac{2}{3}\left[8 - 3\sqrt{3} - 2\sqrt{2} + 1\right] = \frac{2}{3}(9 - 3\sqrt{3} - 2\sqrt{2})$ 0.5

4. [4 points] Evaluate $\iint_R e^{x/y}\, dA$, where R is the region bounded by the curves $y = x, y = 1, x = 0$.



Type II:

$0 \le x \le y$

$0 \le y \le 1$

• It is easier to integrate w.r.t x

So Describe R as a region of Type II

$\iint_R e^{x/y}\, dA = \int_0^1 \int_0^y e^{x/y}\, dx\, dy$ 2

$= \int_0^1 y e^{x/y} \Big]_{x=0}^{x=y} dy$ 0.5

$= \int_0^1 ye - y \; dy$ 0.5

$= \int_0^1 (e-1)\, y\, dy$

$= (e-1) \cdot \frac{1}{2} y^2 \Big]_0^1$ 0.5

$= (e-1) \cdot \frac{1}{2} = \frac{e-1}{2}$ 0.5

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Math201.04, Quizzes #3 & 4, Term 172

Name:

ID #: **Serial #:**

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1. [6 points] Find the local maximum and minimum values and saddle points of $f(x,y) = x^3 - 12xy + 8y^3$.

- $f_x(x,y) = 3x^2 - 12y$; $f_y(x,y) = -12x + 24y^2$

$$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \Rightarrow \begin{cases} 3x^2 - 12y = 0 \\ -12x + 24y^2 = 0 \end{cases} \Rightarrow \begin{cases} x^2 - 4y = 0 & \text{—— (1)} \\ -x + 2y^2 = 0 & \text{—— (2)} \end{cases}$$ 1

(2) $\Longrightarrow \boxed{x = 2y^2}$ (3)

gives complex roots

$\xrightarrow{(1)} 4y^4 - 4y = 0 \Longrightarrow 4y(y^3 - 1) = 0 \Longrightarrow 4y(y-1)(y^2+y+1) = 0$

$\Longrightarrow y = 0, 1$

ct pts 1.5

- $y = 0 \xrightarrow{(3)} x = 0 \Longrightarrow \boxed{(x,y) = (0,0)}$
- $y = 1 \xrightarrow{(3)} x = 2 \Longrightarrow \boxed{(x,y) = (2,1)}$

$\in$ domain of f

- f_x & f_y exist at all pts (x,y)

- Test: $f_{xx}(x,y) = 6x$, $f_{yy}(x,y) = 48y$; $f_{xy}(x,y) = -12$

$D(x,y) = f_{xx} f_{yy} - (f_{xy})^2 = (6x)(48y) - (-12)^2 = 144(2y - 1)$ 2

0.5 . $D(0,0) = -144 < 0 \Longrightarrow f$ has a saddle pt at $(0,0)$

1 . $D(2,1) = 144 > 0$ & $f_{xx}(2,1) = 12 > 0 \Longrightarrow f$ has a local min at $(2,1)$

. The local min. value of f is $f(2,1) = 8 - 24 + 8 = -8$

2. [6 points] Find the extreme values of $\cancel{f(x,y)}$ $f(x,y,z)$ $= x + 2y + z$ subject to the constraints $x^2 + y^2 = 8$ *and* $y + z = 2$.

We Use Lagrange Multipliers.

Here, $g(x,y,z) = x^2+y^2-8$, $h(x,y,z) = y+z-2$, we solve the system:

$$\begin{cases} f_x = \lambda g_x + \mu h_x \\ f_y = \lambda g_y + \mu h_y \\ f_z = \lambda g_z + \mu h_z \\ g = 0 \\ h = 0 \end{cases} \Longrightarrow \begin{cases} 1 = \lambda(2x) + 0 & \text{—— (1)} \\ 2 = \lambda(2y) + \mu(1) & \text{—— (2)} \\ 1 = 0 + \mu(1) & \text{—— (3)} \\ x^2+y^2-8=0 & \text{—— (4)} \\ y+z-2=0 & \text{—— (5)} \end{cases}$$

1.5

- $(3) \Longrightarrow \mu = 1 \xrightarrow{(2)} 2 = 2\lambda y + 1 \Longrightarrow 2\lambda y = 1 \Longrightarrow \boxed{y = \frac{1}{2\lambda}}$ (6)
- $(1) \Longrightarrow \boxed{x = \frac{1}{2\lambda}}$ (7)
- $(7) \,\&\, (8) \xrightarrow{(4)} \frac{1}{4\lambda^2} + \frac{1}{4\lambda^2} - 8 = 0 \Longrightarrow \frac{1}{2\lambda^2} = 8 \Longrightarrow \lambda^2 = \frac{1}{16} \Longrightarrow \lambda = \pm\frac{1}{4}$

0.5

- $\lambda = \frac{1}{4} \xrightarrow{6,7} x = 2,\ y = 2 \xrightarrow{(5)} z = 2-y = 2-2 = 0$

  we get the point $\boxed{(x,y,z) = (2,2,0)}$

1

- $\lambda = -\frac{1}{4} \xrightarrow{6,7} x = -2,\ y = -2 \xrightarrow{(5)} z = 2-y = 2+2 = 4$

  we get the point $\boxed{(x,y,z) = (-2,-2,4)}$

1

- $f(2,2,0) = 2+4+0 = 6$

  $f(-2,-2,4) = -2-4+4 = -2$

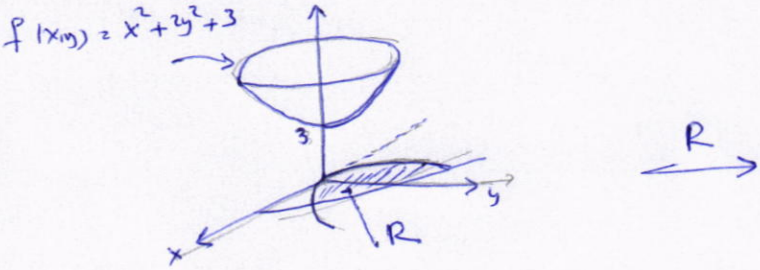
- The max. value of f is 6 & it occurs at the point $(2,2,0)$

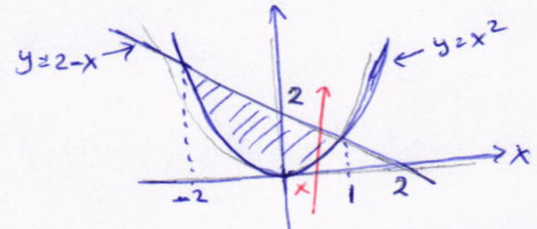
1

- , — min ———— -2 ———— $(-2,-2,4)$

1

3. [4 points] Set up an integral for the volume of the solid that lies below the graph of $z = x^2 + 2y^2 + 3$ and above the region R bounded by the curves $y = x^2$ *and* $y = 2 - x$. **Do not evaluate the integral.**





pts of intersection

$x^2 = 2 - x$

$x^2 + x - 2 = 0$

$(x+2)(x-1) = 0$

$x = -2, \ x = 1$

1

$V = \iint_R f(x,y)\, dA$ 0.5

$= \int_{-2}^{1} \int_{x^2}^{2-x} (x^2+2y^2+3)\, dy\, dx$ 2.5

1+1+0.5

R : Type I

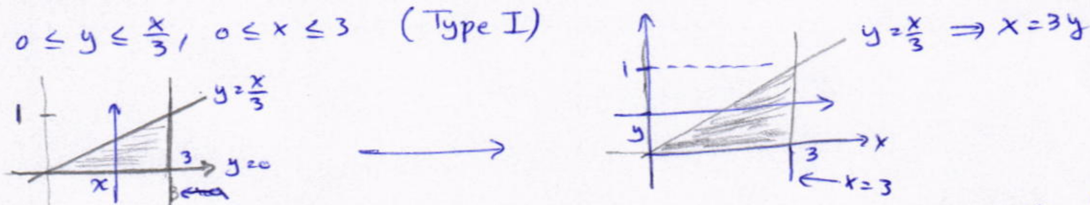
$x^2 \le y \le 2-x$

$-2 \le x \le 1$

4. [4 points] Evaluate $\int_0^3 \int_0^{x/3} \sin(y^2)\, dy\, dx$.

• Not easy to Integrate. So Reverse the order of integration.

• R : $0 \le y \le \frac{x}{3}$, $0 \le x \le 3$ (Type I)



R : $3y \le x \le 3$, $0 \le y \le 1$ (Type I)

$\int_0^3 \int_0^{x/3} \sin(y^2)\, dy\, dx = \int_0^1 \int_{3y}^{3} \sin(y^2)\, dx\, dy$ 2

$= \int_0^1 \sin(y^2) \cdot x \Big]_{x=3y}^{x=3} dy$ 0.5

$= \int_0^1 \sin(y^2)\,(3-3y)\, dy$ 0.5

$= \int_0^1 3\sin(y^2)\, dy - 3\int_0^1 y \sin(y^2)\, dy$ 0.5

$= \int_0^1 3\sin(y^2)\, dy + \frac{3}{2}\cos(y^2)\Big]_0^1 = \int_0^1 3\sin(y^2)\, dy + \frac{3}{2}(\cos 1 - 1)$

0.5/