Midterm Exam for Math 690, Semester 171
Duration: two hours

In this exam, $\Omega \subset \mathbb{R}^2$ is a bounded convex polygonal domain with boundary $\partial\Omega$, $(\cdot,\cdot)$ denotes the $L^2(\Omega)$ inner product and $\|\cdot\|$ is the associated norm.

For $\alpha > 0$, $\mathcal{I}^\alpha$ is the RL time fractional integral of order α,

$$\mathcal{I}^\alpha \varphi(t) = \int_0^t \omega_\alpha(t-s)\varphi(s)\,ds \quad \text{with} \quad \omega_\alpha(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \quad t > 0,$$

and D^α is the RL time fractional derivative,

$$\mathrm{D}^\alpha \varphi(t) := \frac{d^n}{dt^n}\mathcal{I}^{n-\alpha}\varphi(t), \quad n-1 < \alpha < n, \quad t > 0.$$

Q1. Let $g \in C^2[0,b]$ with $g(0) = 0$. Write D^γ in terms of the Caputo derivative, where $1 < \gamma < 2$.

Q2. Let $\beta > 0$.
a) Verify (in details) that

$$\frac{d}{dt}\mathcal{I}^{\beta+1}\varphi(t) = \mathcal{I}^\beta\varphi(t),$$

b) Solve the following fractional integral equation:

$$w'(t) + 3\mathcal{I}^\beta w(t) = 0, \quad t \in (0,T], \quad \text{with} \ \ w(0) = 2.$$

Q3. Consider the following problem: for $0 < \alpha < 1$,

$$w'(t) + D^{1-\alpha}w(t) = f(t), \quad t \in (0,T], \quad \text{with} \ \ w(0) = 0.$$

Let $t_j = j\delta$ $(0 \le j \le N)$ where $\delta = T/N$. Let $W^j \approx w^j := w(t_j)$ (for $1 \le j \le N$) with $W^0 = 0$, be the implicit finite difference (backward Euler in time) solution, defined by

$$W^n - W^{n-1} + \int_{t_{n-1}}^{t_n} D^{1-\alpha}\overline{W}(t)\,dt = \delta\, f(t_n), \quad \text{for } 1 \le n \le N,$$

where $\overline{W}(t) = W^i$ for $t_{i-1} < t \le t_i$, and for $1 \le i \le N$.

a) Show that

$$|W^n| \le 2\delta \sum_{j=1}^{n} |f(t_j)|, \quad \text{for } 1 \le n \le N.$$

b) Find W^n in terms of $f(t_n)$ and W^j for $1 \le j \le n-1$.
c) State the expected convergence rates of the above scheme.

Q4. Consider the following problem

$$\begin{aligned} &u'(x,t) - \mathrm{D}^{1-\alpha}\Delta u(x,t) = f(x,t) && \text{in } \Omega \times (0,T], \\ &u(x,t) = 0 && \text{on } \partial\Omega \times (0,T], \\ &u(x,0) = 0 && \text{in } \Omega, \end{aligned}$$

where $0 < \alpha < 1$ and f is a smooth function.

For each $t \in (0,T]$, let $u_h(t)$ be the (piecewise linear) Galerkin finite element approximation (in space) of the exact solution $u(t)$ with $u_h(0) = 0$.

a) Define $u_h(t)$ in details.
b) Write the scheme in a matrix form.
c) Show the stability of Galerkin finite element solution $u_h(t)$.
d) State the expected convergence rates of the Galerkin finite element scheme under consideration.
e) Replace the homogenous Dirichlet boundary condition with the homogeneous Neumann boundary condition in the above model problem, then define $u_h(t)$.