King Fahd University of Petroleum & Minerals
Department of Mathematics & Statistics

Math 566 (171)
Fractional Differential Equations

Instructor: Khaled Furati

Assignment 1

1) Let $m, n \in \mathbb{N}$. Express $(m-n+1)_n$ in terms of the factorial function.
2) If $f \in C[0,1]$ and $I_0^{\sqrt{2}} f$ is constant on (0,1), find f. Is it unique?
3) Find the most general continuous function that satisfies $D_0^{\alpha} u(t) = 0$, $t > 0$.
4) Calculate $D_1^{3/2}\left[t(t-1)^{1/4}\right]$.

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Assignment 2

Let $y(x) = x^3$ and

$$w(x) = \begin{cases} y(x), & 0 < x < 1, \\ 0\ , & \text{otherwise.} \end{cases}$$

1. Write explicitly the difference $(\Delta_h^3 + \Delta_{-h}^3)\, y$ and compare it with y'''.
2. For $\alpha > 0$, $\alpha \notin \mathbb{N}$, write the first 4 terms of each of the differences: $\Delta_h^{\alpha} y$, $\Delta_{-h}^{\alpha} y$, $\Delta_{h,0+}^{\alpha} y$, and $\Delta_{h,1-}^{\alpha} y$.
3. Use these differences to approximate $\frac{d^{\alpha} w}{d|x|^{\alpha}}$ at $x = 0.6$ when $\alpha = \sqrt{2}$ and $h = 0.1$.

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Assignment 3

Use the series expansions to show the following:

1. $E_{m,\beta-m}(t^m) = t^m\, E_{m,\beta}(t^m)$, $\beta = 0,1,2,\dots,m$.
2. $D_0^{\alpha}\left[\cosh\left(\sqrt{\lambda}\, t\right)\right] = t^{-\alpha}\, E_{2,1-\alpha}(\lambda t^2)$, $\alpha > 0$.
3. $y(t) = E_{2,1}(t^2)$ satisfy $y'' - y = 0$.
4. Use **`mlf`** Matlab function to plot $E_{0.5,1.2}\left(\sqrt{t}\right)$. Is the function concave up? Is it completely monotone?

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Assignment 4

Consider the equation

$$D^{\alpha}y = 2\,x^{\beta}y, \qquad 1 < \alpha < 2.$$

1. State the formula for the fundamental system of solutions for any $\beta \in \mathbb{R}$.
2. State the formula for the fundamental system of solutions when $\beta = -\alpha$.
3. For $\alpha = -\beta = 4/3$, by direct substitution into the equation (not formula), find $\nu > 0$ such that x^{ν} is a solution. Compare with the solutions in parts (1) and (2). (Hint. Graph of Gamma function could be used.)
4. For $\beta = 0$, use Laplace transform to find the fundamental system of solutions. Compare with the solutions in part (1).

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Assignment 5

Note: Section and equation references are as in Kilbas et al. χ is the characteristic function.

Problem 1. (7.6.1 Dynamics of a Sphere Immersed in an Incompressible Viscous Fluid)

A. Provide a one-page description of the physical problem containing the following items.
   - Diagram
   - Dynamics and quantities under consideration
   - General model considered and its solution

B. Consider the following model problem:

$$ {}_0^C D_t^{\alpha} y + \mu\, {}_0^C D_t^{\beta} y + \lambda y = f(t), \qquad \mu > 0, \qquad 0 < \beta < \alpha < 1,$$
$$ y(0) = d.$$

   Use Laplace transform to solve this problem and compare with the solution for (7.6.12) provided by Kilbas et al.

C. Plot in one figure the solutions for $\alpha = 0.6, 0.8, 1.0$, that correspond to

$$d = 0, \qquad \beta = 0.5, \qquad f(t) = 8\,\chi_{[0,1]}, \qquad \mu = \lambda = 1.$$

   Describe the dynamics.

D. Plot in one figure the solutions for $\beta = 0.5, 0.7, 0.9$, that correspond to

$$d = 0, \qquad \alpha = 1.0, \qquad f(t) = 8\,\chi_{[0,1]}, \qquad \mu = \lambda = 1.$$

   Describe the dynamics.

Problem 2. (7.6.2 Oscillatory Processes with Fractional Damping (Bagley-Torvik equations))

A. Provide a one-page description of the physical problem as in Problem 1.

B. Consider the following model problem:

$$ {}_0^C D_t^{\alpha} y + \mu\, {}_0^C D_t^{\beta} y + \lambda y(t) = f(t), \qquad 1 < \beta < \alpha \le 2,$$
$$ y(0) = d_0, \qquad y'(0) = d_1.$$

   Use Laplace transform to solve this problem and compare with the solution (7.6.36) for (7.6.19).

C. Plot in one figure the solutions for $\alpha = 1.6, 1.8, 2.0$ that correspond to

$$d_0 = d_1 = 0, \qquad \beta = 1.5, \qquad f(t) = 2\,\chi_{[0,1]}, \qquad \mu = \lambda = 0.5.$$

   Describe the oscillatory process. See Figure 8.4 by Podlubny.

D. Plot in one figure the solutions for $\beta = 1.5, 1.7, 1.9$, that correspond to

$$d_0 = d_1 = 0, \qquad \alpha = 2.0, \qquad f(t) = 2\,\chi_{[0,1]}, \qquad \mu = \lambda = 1.$$

   Describe the oscillatory process.

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Assignment 6 Talk Report (max. 3 pages)

Prepare a report on your selected talk that includes the following:

- Topic: description, motivation, significance, methods
- Results: clarity of the results and appropriateness of the interpretations
- Future work, recommendations, and extensions suggested by the speaker
- Relevance of the talk to the course, and whether the talk enhanced your learning
- What did you like the most and the least?
- Any other comments, observations, …