King Fahd University of Petroleum & Minerals
Department of Mathematics and Statistics

Math 551 Abstract Algebra - Term 171
Final Exam (Duration = **6 hours**)

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(1) [6 points]
(a) Show that a divisible module over a PID is injective.
(b) Let R be a ring (not necessarily commutative). Prove that $Hom_{\mathbb{Z}}(R, G)$ is an injective (left) R-module for any divisible Abelian group G.
(c) Use the fact "*Every Abelian group can be embedded in a divisible abelian group*" to prove that every (left) R-module can be embedded in an injective (left) R-module."

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(2) [7 points] Let R be an integral domain and let K denote its quotient field. Prove:
(a) K is an injective R-module.
(b) Every K-vector space is an injective R-module.

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(3) [6 points] Let R be a Noetherian ring and M an R-module.
Let $Supp(M) := \{p \in Spec(R) : M_p \neq 0\}$.
(a) Let $x \in M$ and $p \in Spec(R)$. Show: $(Rx)_p \neq 0 \Leftrightarrow Ann(x) \subseteq p$.
(b) Let $a \in R$ and $a_M: M \to M, x \to ax$. Prove: a_M locally nilpotent $\Leftrightarrow a \in \bigcap_{p \in \mathrm{Supp(M)}} p$
(c) Assume that M is finitely generated. Prove: $\sqrt{\mathrm{Ann(M)}} = \bigcap_{p \in \mathrm{Supp(M)}} p$.
(d) Apply **(c)** to deduce a well-known result on Nilradical of R.

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(4) [6 points] Let R be a commutative Artinian ring; that is, R satisfies the descending chain condition (dcc).
(a) Prove that R satisfies the minimum condition; that is, every nonempty set of ideals of R has a minimal element.
(b) Prove that the nilradical of R is nilpotent.

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(5) [8 points] A commutative ring is ***quasi-Frobenius*** if it is Noetherian and injective as a module over itself. Let K be a field. A (commutative) finite-dimensional K-algebra R is called a ***Frobenius algebra*** if R is isomorphic to its K-vector space dual $R^* = Hom_K(R, K)$ as R-modules.
(a) Prove that every Frobenius algebra is quasi-Frobenius.
(b) Let R be a (commutative) finite-dimensional K-algebra. Prove: If there is $f \in R^*$ such that $Ker(f)$ contains no nonzero ideals, then R is a Frobenius algebra.
(c) Deduce from above: If G is a finite (Abelian) group, then the group ring $K[G]$ is quasi-Frobenius.

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(6) [6 points] Recall that a ring R is semisimple if it is semisimple as an R-module. Prove that the following conditions are equivalent for a ring R:
(i) R is semisimple;
(ii) Every (left) R-module is semisimple;
(iii) Every (left) R-module is injective;
(iv) Every (left) R-module is projective.

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(7) [6 points] Let R be a semisimple ring with $I_1, \ldots, I_s$ its non-isomorphic simple (left) ideals. Let E be a nonzero R-module. Prove that $E = \bigoplus_{1 \le i \le s} E_i$ where E_i = Sum of all simple submodules of E isomorphic to I_i for $i = 1, \ldots, s$.