King Fahd University of Petroleum and Minerals,

Department of Mathematics and Statistics- Term 171

Final Exam : Math 550, Linear Algebra

Duration: 4 Hours

NAME :

ID :

Exercise 1. (5-5-5)

Let V be an n-vector space over a field F and T be a linear operator on V.

(1) Assume that $Nullspace(T)$ is isomorphic to $range(T)$. Prove that $n = dimV$ is even.

(2) Find an example of a vector space V and T a linear operator on V such that $Nullspace(T)$ is isomorphic to $range(T)$.

(3) Assume that $range(T) \cap Nullspace(T) = \{0\}$ and $T^2 = 0$. Prove that $T = 0$.

Exercise 2. (5-5-5-5 points)

Let $A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & a & 2 \end{pmatrix}$ where a is a real number.

(1) Find the characteristic polynomial and the minimal polynomial of A.

(2) For which values of a the matrix A is in the Jordan form?

(3) Given two distinct values $a_1 \neq a_2$ of a where A is in the Jordan form. Do the matrices A_1 (for $a = a_1$) and A_2 (for $a = a_2$) are similar? Justify.

(4) Give an example of two matrices M and N with the same characteristic polynomials and same minimal polynomials that are not similar.

Exercise 3. (5-5-5)

Let V be an n-dimensional vector space over a field F, T a linear operator on V and N a Nilpotent linear operator on V.

(1) Assume that $N^{n-1}\alpha \neq 0$ for some $\alpha \in V$. Prove that α is a cyclic vector for N.

(2) Assume that T^2 has a cyclic vector. Prove that T has a cyclic vector.

(3) If T has a cyclic vector, does T^2 has a cyclic vector?

Exercise 4. (15 points)

Let V be an n-dimensional complex inner product space, T a linear operator of V and E an idempotent linear operator on V.

(1) Prove that E is self-adjoint if and only if E is normal.

(2) Prove that T is self adjoint if and only if $(T\alpha|\alpha)$ is a real number for every $\alpha \in V$.

Exercise 5. (15 points)

Let $B = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

(1) Apply Gram-Schmidt process to the rows of B to obtain an orthonormal ordered basis $B' = \{\alpha_1, \alpha_2, \alpha_3\}$.

(2) Use (1) to find a unitary matrix U and a matrix M such that $U = MB$.

Exercise 6. (5-5-5)

Let V be a complex inner product space and T a linear operator on V.

Prove that the following assertions are equivalent.

(1) T is normal;

(2) $||T(\alpha)|| = ||T^*(\alpha)||$ for every $\alpha \in V$.

(3) $T = T_1 + iT_2$ where T_1, T_2 are self-adjoint operators on V and $T_1T_2 = T_2T_1$.

Exercise 7. (15 points)

Let V be a inner product space over a field F, T a normal operator on V.

(1) Prove that for every polynomial $f(X) \in F[X]$, $f(T)$ is a normal operator on V.

(2) Let $f(X)$ and $g(X)$ be two polynomials in $F[X]$ that are relatively prime and suppose that there is α and β in V such that $f(T)(\alpha) = g(T)(\beta) = 0$. Prove that $(\alpha|\beta) = 0$.

Exercise 8. (5-5-5-5)

Let $V = \mathbb{R}^3$, $S = \{e_1, e_2, e_3\}$ its standard basis and f the skew symmetric bilinear form on V defined by $f(X,Y) = x_1y_2 - x_1y_3 - x_2y_1 + 2x_2y_3 + x_3y_1 - 2x_3y_2$.

(1) Find $[f]_S$.

(2) Find $rank(f)$.

(3) Let $W = span\{e_1, e_2\}$. Find a basis B for $W^\perp$.

(4) Find $[f]_{B'}$ where B' is the basis $B' = \{e_1, e_2\} \cup B$.