King Fahd University of Petroleum & Minerals
Department of Mathematics and Statistics

MATH 302, Semester 171 (2017-2018)

EXAM I
October 25, 2017

Allowed Time: 120 mins

Student Name:

Student ID Number:

Section Number:

Instructions:
1. Write neatly and legibly -- *you may lose points for messy work.*
2. Show all your work -- *no points for answers without justification.*
3. Programmable calculators and Mobiles are **not** allowed.
4. Make sure that you have 6 problems (6 pages + cover page + Scratch sheet).

Problem No.	Points	Maximum Points
1		10
2		15
3		10
4		15
5		25
6		25
Total:		**100**

Problem 1. Show that the vectors $U =< -1,0,2,3 >$, $V =< 0,0,-2,4 >$, $W =< -2,0,2,10 >$ are linearly dependent

For $\alpha, \beta, \gamma \in \mathbb{R}$,

$\alpha U + \beta V + \gamma W = 0 \Rightarrow$

$< -\alpha, 0, 2\alpha, 3\alpha > + < 0, 0, -2\beta, 4\beta > + < -2\gamma, 0, 2\gamma, 10\gamma >$

$= < 0, 0, 0, 0 >$

So, we have

$< -\alpha - 2\gamma, 0, 2\alpha - 2\beta + 2\gamma, 3\alpha + 4\beta + 10\gamma > = < 0, 0, 0, 0 >$

$\Rightarrow \quad \alpha + 2\gamma = 0 \qquad (1)$

$2\alpha - 2\beta + 2\gamma = 0 \qquad (2)$

$3\alpha + 4\beta + 10\gamma = 0 \qquad (3)$

From (1) $\alpha = -2\gamma$

From (2) $\beta = \alpha + \gamma = -\gamma$

If $\gamma = 1$, then $\alpha = -2$, $\beta = -1$

replacing in (3), we have $-6 - 4 + 10 = 0$ (satisfied)

Therefore, we have

$-2U - V + W = 0$

$\Rightarrow$ U, V, W are linearly dependent.

Problem 2. **(a)** Check that the set

$$S = \{X = < x_1, x_2, x_3, x_4 > \in \mathbb{R}^4 : x_1 = x_2 = x_3 = x_4\}$$

is a subspace of $\mathbb{R}^4$. Find a basis and the dimension of S.

1°) let $X = < x, x, x, x >$ and $Y = < y, y, y, y > \in S$

Then $X + Y = < x+y, x+y, x+y, x+y >$

$= < z, z, z, z > \ / \ z = x+y$

$\therefore X + Y \in S$

2°) If $\alpha \in \mathbb{R}$ then $\alpha X = < \alpha x, \alpha x, \alpha x, \alpha x >$

then $\alpha X \in S$.

So S is a subspace.

<u>Basis</u> If $X \in S$, then $X = < x, x, x, x >$

$= x < 1, 1, 1, 1 >$

A basis is $\{ < 1, 1, 1, 1 > \}$

$\dim S = 1$.

(b) Is the set $F = \{X = < x, y > \in \mathbb{R}^2 : y = x^2\}$ a subspace of $\mathbb{R}^4$? Justify your answer

let $a = < 1, 1 > \in F$, $b = < 2, 4 > \in F$

But $a + b = < 3, 5 > \notin F$ since $5 \neq 3^2$

Problem 3. Use the Gauss method to find all solutions of the system

$$\begin{cases} -x+y+\quad 2w=0 \\ \quad\;\; y-z+w=1 \\ -x\quad\;\; +z+w=0 \end{cases}$$

<u>Sol.</u>

$$\hat{A}=\left(\begin{array}{cccc|c} \boxed{-1} & 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 1 & 1 \\ -1 & 0 & 1 & 1 & 0 \end{array}\right) \xrightarrow[R_3-R_1]{-R_1} \left(\begin{array}{cccc|c} 1 & -1 & 0 & -2 & 0 \\ 0 & \boxed{1} & -1 & 1 & 1 \\ 0 & -1 & 1 & -1 & 0 \end{array}\right)$$

$$\xrightarrow[R_3+R_2]{R_1+R_2} \left(\begin{array}{cccc|c} 1 & 0 & -1 & -1 & 1 \\ 0 & 1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)=\hat{A}_R$$

$\dim \hat{A}=3$, $\dim A=2$

So the system is inconsistent

No solution.

Problem 4. According to the values of $m \in \mathbb{R}$, find the rank of the matrix

$$A = \begin{pmatrix} 1 & m & m \\ m & 1 & m \\ m & m & 1 \end{pmatrix}$$

Sol. Case $m = 1$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \xrightarrow[R_3 - R_1]{R_2 - R_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

So rank $A = 1$.

Case $m \neq 1$, we do some reduction

$$\begin{pmatrix} 1 & m & m \\ m & 1 & m \\ m & m & 1 \end{pmatrix} \xrightarrow[R_3 - R_2]{R_2 - mR_1} \begin{pmatrix} 1 & m & m \\ 0 & 1-m^2 & m-m^2 \\ 0 & m-1 & 1-m \end{pmatrix} \xrightarrow[R_2/(m-1)]{R_2/(1-m)}$$

$$\begin{pmatrix} 1 & m & m \\ 0 & 1+m & m \\ 0 & 1 & -1 \end{pmatrix} \xrightarrow{R_2 \to R_3} \begin{pmatrix} 1 & m & m \\ 0 & \boxed{1} & -1 \\ 0 & 1+m & m \end{pmatrix} \xrightarrow[R_2 - (1+m)R_2]{R_1 - mR_2}$$

$$\begin{pmatrix} 1 & 0 & 2m \\ 0 & 1 & -1 \\ 0 & 0 & 2m+1 \end{pmatrix}.$$

So, if $m = -\frac{1}{2}$ then rank $A = 2$

If $m \neq -\frac{1}{2}$ and $m \neq 1$ then rank $A = 3$

Problem 5. (a) Find <u>all real values</u> of b, for which the matrix $B = \begin{pmatrix} 1 & 2 \\ 3 & b \end{pmatrix}$ has complex conjugate eigenvalues

$$\begin{vmatrix} 1-\lambda & 2 \\ 3 & b-\lambda \end{vmatrix} = \lambda^2 - (1+b)\lambda + b - 6 = 0$$

So, in order to have complex conjugate eigenvalues we have to have $\Delta = (1+b)^2 - 4(b-6) < 0$

That is, $b^2 - 2b + 25 = (b-1)^2 + 24 < 0$

But this is impossible $\forall\, b \in \mathbb{R}$.

Thus, B has <u>no</u> complex eigenvalues

(b) Find all the eigenvalues of the matrix

$$A = \begin{pmatrix} 3 & 2 & -1 \\ 3 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 2 & -1 \\ 3 & -1-\lambda & 0 \\ 1 & 0 & -1-\lambda \end{vmatrix}$$

$$= \begin{vmatrix} 2 & -1 \\ -1-\lambda & 0 \end{vmatrix} + (-1-\lambda)\begin{vmatrix} 3-\lambda & 2 \\ 3 & -1-\lambda \end{vmatrix}$$

$$= -(1+\lambda) - (1+\lambda)(\lambda^2 - 2\lambda - 9)$$

$$= -(1+\lambda)(\lambda^2 - 2\lambda - 8) = -(\lambda+1)(\lambda+2)(\lambda-4) =$$

Therefore, $\lambda = -2, -1, 4$ are the eigenvalues

(b) Without any further computation, say why A is diagonalizable and write down the corresponding diagonal matrix D.

Since the eigenvalues are distinct then A is diagonalizable

$$D = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

Problem 6. Given the matrix $M = \begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix}$

(a) Find an orthogonal matrix P which diagonalizes M; that is $P^{-1}MP = D$

$$\begin{vmatrix} 3-\lambda & -2 \\ -2 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 - 4 = (\lambda-1)(\lambda-5) = 0$$

$\Rightarrow \lambda = 1, 5$ are the eigenvalues

Eigenvectors

$\boxed{\lambda = 1}$ $\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \Rightarrow x_1 = x_2$

So $E_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector

$\boxed{\lambda = 5}$ $\begin{pmatrix} -2 & -2 \\ -2 & -2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \Rightarrow x_1 = -x_2$

$\Rightarrow E_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector

An orthogonal matrix is $P = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$

(b) Write down P^{-1} and D.

$P^{-1} = P^T = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \end{pmatrix}$, $D = \begin{pmatrix} 1 & 0 \\ \end{pmatrix}$