

King Fahd University of Petroleum & Minerals
Department of Mathematics and Statistics

MATH 302, Semester 171 (2017-2018)

EXAM I
October 25, 2017

Allowed Time: 120 mins

Student Name:

Student ID Number:

Section Number:

Instructions:

1. Write neatly and legibly -- *you may lose points for messy work.*
2. Show all your work -- *no points for answers without justification.*
3. Programmable calculators and Mobiles are **not** allowed.
4. Make sure that you have 6 problems (6 pages + cover page + Scratch sheet).

Problem No.	Points	Maximum Points
1		10
2		15
3		10
4		15
5		25
6		25
Total:		100

Problem 1. Show that the vectors $U = \langle -1, 0, 2, 3 \rangle$, $V = \langle 0, 0, -2, 4 \rangle$,
 $W = \langle -2, 0, 2, 10 \rangle$ are linearly dependent

For $\alpha, \beta, \gamma \in \mathbb{R}$,

$$\alpha U + \beta V + \gamma W = 0 \Rightarrow$$

$$\langle -\alpha, 0, 2\alpha, 3\alpha \rangle + \langle 0, 0, -2\beta, 4\beta \rangle + \langle -2\gamma, 0, 2\gamma, 10\gamma \rangle \\ = \langle 0, 0, 0, 0 \rangle$$

So, we have

$$\langle -\alpha - 2\gamma, 0, 2\alpha - 2\beta + 2\gamma, 3\alpha + 4\beta + 10\gamma \rangle = \langle 0, 0, 0, 0 \rangle$$

$$\Rightarrow \alpha + 2\gamma = 0 \quad (1)$$

$$2\alpha - 2\beta + 2\gamma = 0 \quad (2)$$

$$3\alpha + 4\beta + 10\gamma = 0 \quad (3)$$

From (1) $\alpha = -2\gamma$

From (2) $\beta = \alpha + \gamma = -\gamma$

If $\gamma = 1$, then $\alpha = -2$, $\beta = -1$

replacing in (3), we have $-6 - 4 + 10 = 0$ (satisfied)

Therefore, we have

$$-2U - V + W = 0$$

$\Rightarrow U, V, W$ are linearly dependent.

Problem 2. (a) Check that the set

$$S = \{X = \langle x_1, x_2, x_3, x_4 \rangle \in \mathbb{R}^4 : x_1 = x_2 = x_3 = x_4\}$$

is a subspace of \mathbb{R}^4 . Find a basis and the dimension of S .

10) let $X = \langle x, x, x, x \rangle$ and $Y = \langle y, y, y, y \rangle \in S$

$$\begin{aligned} \text{Then } X + Y &= \langle x+y, x+y, x+y, x+y \rangle \\ &= \langle z, z, z, z \rangle \quad | \quad z = x+y \end{aligned}$$

$$\therefore X + Y \in S$$

20) If $\alpha \in \mathbb{R}$ then $\alpha X = \langle \alpha x, \alpha x, \alpha x, \alpha x \rangle$
then $\alpha X \in S$.

So S is a subspace.

Basis If $X \in S$, then $X = \langle x, x, x, x \rangle$
 $= x \langle 1, 1, 1, 1 \rangle$

A basis is $\{ \langle 1, 1, 1, 1 \rangle \}$

$$\dim S = 1.$$

(b) Is the set $F = \{X = \langle x, y \rangle \in \mathbb{R}^2 : y = x^2\}$ a subspace of \mathbb{R}^2 ?
Justify your answer

let $a = \langle 1, 1 \rangle \in F$, $b = \langle 2, 4 \rangle \in F$

But $a + b = \langle 3, 5 \rangle \notin F$ since $5 \neq 3^2$

Problem 3. Use the Gauss method to find all solutions of the system

$$\begin{cases} -x + y + 2w = 0 \\ y - z + w = 1 \\ -x + z + w = 0 \end{cases}$$

Sol.

$$\hat{A} = \left(\begin{array}{cccc|c} \boxed{-1} & 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 1 & 1 \\ -1 & 0 & 1 & 1 & 0 \end{array} \right) \xrightarrow[\substack{-R_1 \\ R_3 - R_1}]{} \left(\begin{array}{cccc|c} 1 & -1 & 0 & -2 & 0 \\ 0 & \boxed{1} & -1 & 1 & 1 \\ 0 & -1 & 1 & -1 & 0 \end{array} \right)$$

$$\xrightarrow[\substack{R_1 + R_2 \\ R_3 + R_2}]{} \left(\begin{array}{cccc|c} 1 & 0 & -1 & -1 & 1 \\ 0 & 1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right) = \hat{A}_R$$

$$\dim \hat{A} = 3, \quad \dim A = 2$$

So the system is inconsistent
No solution.

Problem 4. According to the values of $m \in \mathbb{R}$, find the rank of the matrix

$$A = \begin{pmatrix} 1 & m & m \\ m & 1 & m \\ m & m & 1 \end{pmatrix}$$

Sol. Case $m=1$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \xrightarrow[\substack{R_2 - R_1 \\ R_3 - R_1}]{} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

So $\text{rank } A = 1$.

Case $m \neq 1$, we do some reduction

$$\begin{pmatrix} 1 & m & m \\ m & 1 & m \\ m & m & 1 \end{pmatrix} \xrightarrow[\substack{R_2 - mR_1 \\ R_3 - mR_1}]{} \begin{pmatrix} 1 & m & m \\ 0 & 1-m^2 & m-m^2 \\ 0 & m-1 & 1-m \end{pmatrix} \xrightarrow[\substack{R_2/(1-m) \\ R_2/(m-1)}]{} \begin{pmatrix} 1 & m & m \\ 0 & 1 & -1 \\ 0 & m-1 & 1-m \end{pmatrix}$$

$$\begin{pmatrix} 1 & m & m \\ 0 & 1+m & m \\ 0 & 1 & -1 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & m & m \\ 0 & \boxed{1} & -1 \\ 0 & 1+m & m \end{pmatrix} \xrightarrow[\substack{R_1 - mR_2 \\ R_2 - (1+m)R_2}]{} \begin{pmatrix} 1 & 0 & 2m \\ 0 & 1 & -1 \\ 0 & 0 & 2m+1 \end{pmatrix}$$

So, if $m = -\frac{1}{2}$ then $\text{rank } A = 2$

If $m \neq -\frac{1}{2}$ and $m \neq 1$ then $\text{rank } A = 3$

Problem 5. (a) Find all real values of b , for which the matrix $B = \begin{pmatrix} 1 & 2 \\ 3 & b \end{pmatrix}$ has complex conjugate eigenvalues

$$\begin{vmatrix} 1-\lambda & 2 \\ 3 & b-\lambda \end{vmatrix} = \lambda^2 - (1+b)\lambda + b - 6 = 0$$

So, in order to have complex conjugate eigenvalues we have to have $\Delta = (1+b)^2 - 4(b-6) < 0$

$$\text{That is, } b^2 - 2b + 25 = (b-1)^2 + 24 < 0$$

But this is impossible $\forall b \in \mathbb{R}$.

Thus, B has no complex eigenvalues

(b) Find all the eigenvalues of the matrix

$$A = \begin{pmatrix} 3 & 2 & -1 \\ 3 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 2 & -1 \\ 3 & -1-\lambda & 0 \\ 1 & 0 & -1-\lambda \end{vmatrix}$$

$$= \begin{vmatrix} 2 & -1 \\ -1-\lambda & 0 \end{vmatrix} + (-1-\lambda) \begin{vmatrix} 3-\lambda & 2 \\ 3 & -1-\lambda \end{vmatrix}$$

$$= -(1+\lambda) - (1+\lambda)(\lambda^2 - 2\lambda - 9)$$

$$= -(1+\lambda)(\lambda^2 - 2\lambda - 8) = -(\lambda+1)(\lambda+2)(\lambda-4) =$$

Therefore, $\lambda = -2, -1, 4$ are the eigenvalues

(6) Without any further computation, say why A is diagonalizable and write down the corresponding diagonal matrix D .

Since the eigenvalues are distinct then A is diagonalizable

$$D = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

Problem 6. Given the matrix $M = \begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix}$

(a) Find an orthogonal matrix P which diagonalizes M ; that is $P^{-1}MP = D$

$$\begin{vmatrix} 3-\lambda & -2 \\ -2 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 - 4 = (\lambda-1)(\lambda-5) = 0$$

$\Rightarrow \lambda = 1, 5$ are the eigenvalues

Eigenvectors

$$\boxed{\lambda=1} \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \Rightarrow x_1 = x_2$$

So $\bar{E}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector

$$\boxed{\lambda=5} \begin{pmatrix} -2 & -2 \\ -2 & -2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \Rightarrow x_1 = -x_2$$

$\Rightarrow \bar{E}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector

An orthogonal matrix is $P = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$

(b) Write down P^{-1} and D .

$$P^{-1} = P^T = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$$