Name: ID#: Serial #:

---

1. [8pts] Let P, Q, R be statements. Determine (with justification) whether the following statement is a tautology or a contradiction or neither:

$$(P \longrightarrow (Q \longrightarrow R)) \longrightarrow (P \longrightarrow Q).$$

Solution. This problem can be solved using a Truth Table. It can also be solved as follows. First, let us call S the displayed statement above.

- If P, Q, R are all True, then the statement S is True.
- If P is True and Q is False, then $Q \longrightarrow R$ is True (the truth value of R is irrelevant) so that $P \longrightarrow (Q \longrightarrow R)$ is True, but $P \longrightarrow Q$ is False. In this case, S is False.

Since S can be True and can be False, it is neither a tautology nor a contradiction.

---

2. [12pts] (a) Give an example of sets A, B, C such that $B \neq C$ and $B - A = C - A$.

Solution. Take $B = \emptyset$, $A = C = \{0\}$.

(b) Let A, B, C, D be sets such that $A \subseteq B$ and $C \subseteq D$. Prove that $A - D \subseteq B - C$.

Proof. Let $x \in A - D$. Then $x \in A$ and so $x \in B$ (because $A \subseteq B$) and $x \notin D$ and so $x \notin C$ (because $C \subseteq D$), this proves $x \in B - C$. Hence $A - D \subseteq B - C$. ■

---

3. [8pts] Let A, B, C, D be sets. Prove that $(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$.

Proof. Let $(x, y) \in (A \times B) \cup (C \times D)$. Then $(x, y) \in A \times B$ or $(x, y) \in C \times D$.
Suppose $(x, y) \in A \times B$. Then $x \in A$ and so $x \in A \cup C$, and $y \in B$ and so $y \in B \cup D$. This shows that $(x, y) \in (A \cup C) \times (B \cup D)$.
When $(x, y) \in C \times D$, we similarly obtain $(x, y) \in (A \cup C) \times (B \cup D)$.
This proves $(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$. ■

---

4. [12pts] Let x, y, z be real numbers.
(a) Prove that $|x + y + z| \leq |x| + |y| + |z|$.

Proof. Using the triangle inequality (twice), we have $|x + y + z| \leq |x + y| + |z| \leq |x| + |y| + |z|$. ■

(b) Use (a) to prove that $|x - y - z| \geq |x| - |y| - |z|$.

Proof. By (a), $|x - y - z| + |y| + |z| \geq |x - y - z + y + z| = |x|$, so $|x - y - z| \geq |x| - |y| - |z|$. ■

(c) Use (b) to prove that $|x + y + z| \geq |x| - |y| - |z|$.

Proof. Replace y by $-y$ and z by $-z$ in the inequality in (b) to obtain:
$|x + y + z| \geq |x| - |-y| - |-z| = |x| - |y| - |z|$, as required. ■

---

5. [8pts] Let x, y be integers. Prove that if x and y are even, then $\dfrac{x^2 + y^2}{2}$ is even. Is the same conclusion true when x and y are odd? Justify.

Solution. Suppose x, y are even. Then $x = 2h$ and $y = 2k$ for some integers h, k and $\dfrac{x^2 + y^2}{2} = 2(h^2 + k^2)$, which is even since $h^2 + k^2 \in \mathbb{Z}$.

When x and y are odd, then it is not true that $\frac{x^2+y^2}{2}$ is even, take $x=y=1$. [In fact, if $x=2h+1$ and $y=2k+1$ for some integers h,k, then $\frac{x^2+y^2}{2}=\frac{(2h+1)^2+(2k+1)^2}{2}=2(h^2+h+k^2+k)+1$, which is odd.]

---

6. [12pts] Let $a,b,m\in\mathbb{Z}$, $m\neq 0$. Suppose $m|(a+2)$ and $m|(b+2)$.
(a) Prove that $m|(a-b)$.

Proof. We have $a\equiv -2\,(\mathrm{mod}\,m)$ and $b\equiv -2\,(\mathrm{mod}\,m)$, hence $a\equiv b\,(\mathrm{mod}\,m)$. ■

(b) Prove that $m|(ab-4)$.

Proof. From $a\equiv -2\,(\mathrm{mod}\,m)$ and $b\equiv -2\,(\mathrm{mod}\,m)$, we directly obtain $ab\equiv (-2)^2\,(\mathrm{mod}\,m)$. Hence $ab\equiv 4\,(\mathrm{mod}\,m)$. ■