Math201.10, Quiz #1, Term 171

Name: Solutions

ID #: Serial #:

1. **[3 points]** Describe and sketch, with directions, the parametric curve given by

$$x = \ln t, \qquad y = -\sqrt{t}, \qquad 1 \le t \le e^2.$$

2. **[3 points]** Find the equation of the tangent line to the polar curve $r = \sqrt{3}\, \sin\theta$ at the point corresponding to $\theta = \frac{\pi}{3}$.

3. **[4 points]** Find the area of the polar region that lies **inside** the curve $r = 4\cos\theta$ and **to the left of** the curve $r = \sec\theta$.

Good luck,

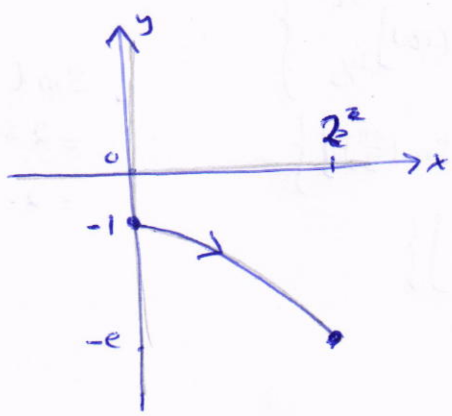
Ibrahim Al-Rasasi

1.5

$\boxed{1}$ • $x = \ln t \Rightarrow t = e^x \Rightarrow y = -\sqrt{e^x} = -e^{x/2}$, an exponential curve

• $1 \le t \le e^2 \Rightarrow \ln 1 \le \ln t \le \ln e^2 \Rightarrow 0 \le x \le 2$

t	(x,y)
1	$(0,-1)$ ← initial
e^2	$(2,-e)$ ← terminal



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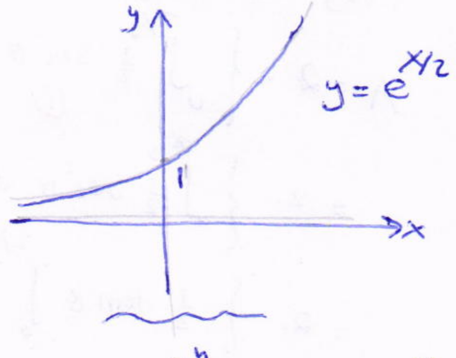
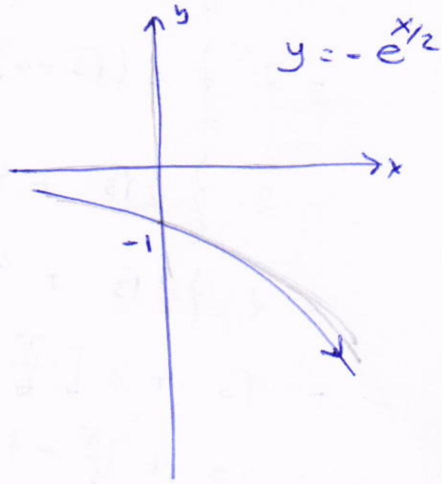




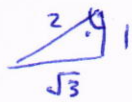
$\boxed{2}$ $r = \sqrt{3}\sin\theta, \quad \theta = \frac{\pi}{3}$ $\qquad r = f(\theta) = \sqrt{3}\sin\theta$

- $\dfrac{dy}{dx} = \dfrac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$

$$= \frac{(\sqrt{3}\cos\theta)\sin\theta + (\sqrt{3}\sin\theta)\cos\theta}{(\sqrt{3}\cos\theta)\cos\theta - (\sqrt{3}\sin\theta)\sin\theta} = \frac{2\cancel{\sqrt{3}}\sin\theta\cos\theta}{\cancel{\sqrt{3}}(\cos^2\theta - \sin^2\theta)}$$

1.5

$$\text{Slope} = \left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{3}} = \frac{2\cdot\frac{\sqrt{3}}{2}\cdot\frac{1}{2}}{(\frac{1}{2})^2 - (\frac{\sqrt{3}}{2})^2} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{4}-\frac{3}{4}} = \frac{\sqrt{3}}{2}\cdot -2 = -\sqrt{3}$$



- $\theta = \frac{\pi}{3} \Rightarrow x = r\cos\theta = (\sqrt{3}\sin\theta)\cos\theta = \sqrt{3}\cdot\frac{\sqrt{3}}{2}\cdot\frac{1}{2} = \frac{3}{4}$

$\qquad y = r\sin\theta = (\sqrt{3}\sin\theta)\sin\theta = \sqrt{3}\cdot\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2} = \frac{3}{4}\sqrt{3}$

1

Eq. of the tangent line is

$$y - \frac{3}{4}\sqrt{3} = -\sqrt{3}\left(x - \frac{3}{4}\right) \Rightarrow y = -\sqrt{3}x + \frac{3\sqrt{3}}{4} + \frac{3}{4}\sqrt{3}$$

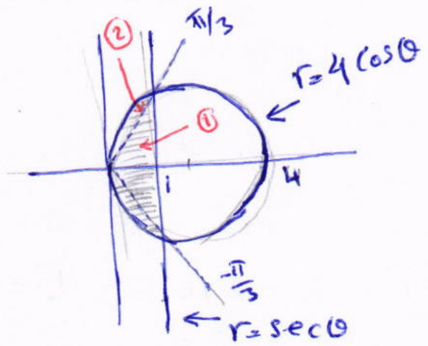
$$\Rightarrow y = -\sqrt{3}x + \frac{3}{2}\sqrt{3}$$

0.5

$\boxed{3}$ · $r = 4\cos\theta$

- $r = \sec\theta \Rightarrow r\cos\theta = 1 \Rightarrow x = 1$ 0.5
- pts of intersection : $4\cos\theta = \sec\theta \Rightarrow \cos^2\theta = \frac{1}{4}$

$\Rightarrow \cos\theta = \pm\frac{1}{2} \Rightarrow \theta = \pm\frac{\pi}{3}$ 0.5



. By symmetry about the x-axis :

$$A = 2\cdot\left\{\int_0^{\pi/3}\frac{1}{2}\sec^2\theta\,d\theta + \int_{\pi/3}^{\pi/2}\frac{1}{2}(4\cos\theta)^2\,d\theta\right\}$$

1 $\qquad$ 2 $\qquad$ 1 $\qquad$ 0.5

$$= 2\cdot\left\{\int_0^{\pi/3}\frac{1}{2}\sec^2\theta\,d\theta + 8\int_{\pi/3}^{\pi/2}\cos^2\theta\,d\theta\right\}$$

$$= 2\cdot\left\{\frac{1}{2}\tan\theta\Big]_0^{\pi/3} + 4\int_{\pi/3}^{\pi/2} 1+\cos(2\theta)\,d\theta\right\}$$

1

$$= 2\cdot\left\{\frac{1}{2}(\sqrt{3}-0) + 4\left[\theta + \frac{1}{2}\sin(2\theta)\right]_{\pi/3}^{\pi/2}\right\}$$

$$= 2\cdot\left\{\frac{1}{2}\sqrt{3} + 4\left[\left(\frac{\pi}{2}+0\right) - \left(\frac{\pi}{3} + \frac{1}{2}\cdot\sin\left(\frac{2\pi}{3}\right)\right)\right]\right\}$$

- $\sin\left(\frac{2\pi}{3}\right) = 2\sin\frac{\pi}{3}\cos\frac{\pi}{3} = 2\cdot\frac{\sqrt{3}}{2}\cdot\frac{1}{2} = \frac{1}{\cancel{2}}\sqrt{3}$

$$= 2\cdot\left\{\frac{1}{2}\sqrt{3} + 4\left[\frac{\pi}{2} - \left(\frac{\pi}{3} + \frac{1}{2}\cdot\frac{\sqrt{3}}{2}\right)\right]\right\}$$

$$= \sqrt{3} + 8\left[\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right]$$

$$= \sqrt{3} + \frac{4\pi}{3} - 2\sqrt{3}$$

$$= \frac{4\pi}{3} - \sqrt{3}$$

0.5

Math201.20, Quiz #1, Term 171

Name:

Solutions

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1. [3 points] Describe and sketch, with directions, the parametric curve given by

$$x = -tant, \qquad y = sect, -\frac{\pi}{2} < t < \frac{\pi}{2}.$$

2. [3 points] Find the length of the parametric curve

$$x = 3t - t^3, \qquad y = 3t^2, \qquad 0 \le t \le 1.$$

3. [4 points] Find the area of the polar region that lies **inside** both curves $r = 1 - cos\theta \; and \; r = 1.$

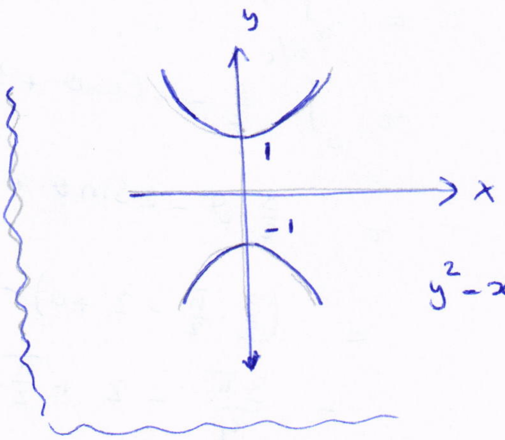
Good luck,

Ibrahim Al-Rasasi

$\boxed{1}$. Since $1+\tan^2 t = \sec^2 t$, then $1+(-x)^2 = y^2$. Thus $1+x^2 = y^2$ or $y^2 - x^2 = 1$, a hyperbola. 1.5

- $-\frac{\pi}{2} < t < \frac{\pi}{2} \Rightarrow y = \sec t \geq 1$,

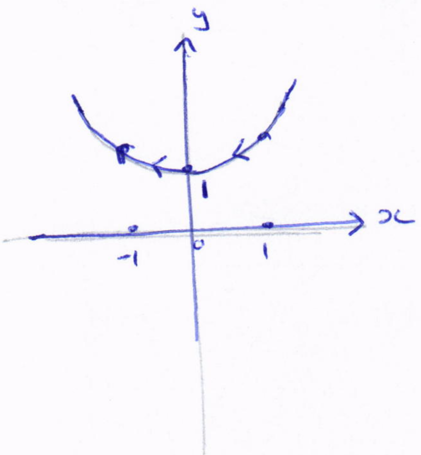
  $\Rightarrow$ we take the upper branch of the hyperbola



$y^2 - x^2 = 1$

- For directions

t	(x, y)
$-\frac{\pi}{4}$	$(1, \sqrt{2})$
0	$(0, 1)$
$\frac{\pi}{4}$	$(-1, \sqrt{2}$



$\boxed{2}$ $x = 3t - t^3$, $y = 3t^2$

$$L = \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^1 3 + 3t^2 \, dt$$

$$= 3t + t^3 \Big]_0^1$$

$$= (3+1) - (0) = 4$$

$\frac{dx}{dt} = 3 - 3t^2$, $\frac{dy}{dt} = 6t$

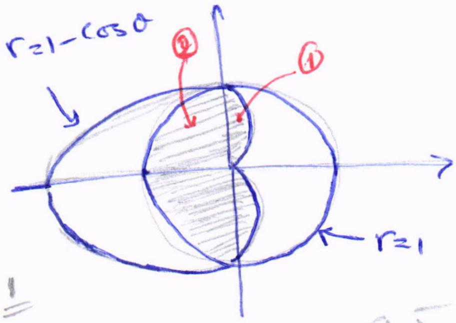
. $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 9 - 18t^2 + 9t^4 + 36t^2$

$= 9 + 18t^2 + 9t^4$

$= (3 + 3t^2)^2$

$\therefore \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(3+3t^2)^2} = |3+3t^2| = 3 + 3t^2$

$\boxed{3}$ $r = 1 - \cos\theta$, $r = 1$



. pts of intersection: $1 - \cos\theta = 1 \Rightarrow \cos\theta = 0$

$\Rightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2}$ 0.5

. By Symmetry about the x-axis:

$$A = 2 \cdot \left\{ \int_0^{\pi/2} \frac{1}{2}(1-\cos\theta)^2 d\theta + \int_{\pi/2}^{\pi} \frac{1}{2}(1)^2 d\theta \right\}$$ 1

① ②

0.5

$$= 2 \left\{ \frac{1}{2} \int_0^{\pi/2} 1 - 2\cos\theta + \cos^2\theta \, d\theta + \frac{1}{2}\int_{\pi/2}^{\pi} d\theta \right\}$$

$$= \int_0^{\pi/2} 1 - 2\cos\theta + \frac{1}{2}(1 + \cos(2\theta))\, d\theta + \int_{\pi/2}^{\pi} d\theta$$ 0.5

$$= \int_0^{\pi/2} \frac{3}{2} - 2\cos\theta + \frac{1}{2}\cos(2\theta)\, d\theta + \int_{\pi/2}^{\pi} d\theta$$

$$= \frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin(2\theta)\Big]_0^{\pi/2} + \theta\Big]_{\pi/2}^{\pi}$$ 1

$$= \left(\frac{3}{2}\cdot\frac{\pi}{2} - 2 + 0\right) - (0) + \left(\pi - \frac{\pi}{2}\right)$$

$$= \frac{3\pi}{4} - 2 + \frac{\pi}{2}$$

$$= \frac{5\pi}{4} - 2.$$ 0.5