

Chapter 3 MQR Markov Chain review

$$p_{ij} = P(X_{n+1} = j | X_n = i) \quad \sum_{j=1}^m p_{ij} = 1 \quad \mathbf{P} = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1m} \\ p_{21} & p_{22} & \cdots & p_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ p_{m1} & p_{m2} & \cdots & p_{mm} \end{bmatrix}$$

$$\boldsymbol{\pi}_n = (\pi_{1n}, \pi_{2n}, \dots, \pi_{mn}) \quad \sum_{i=1}^m \pi_{in} = 1 \quad \boldsymbol{\pi}_{n+r} = \begin{cases} \boldsymbol{\pi}_n \cdot \mathbf{P} \cdot \mathbf{P} \cdots \mathbf{P} = \boldsymbol{\pi}_n \cdot \mathbf{P}^r & \text{homogeneous} \\ \boldsymbol{\pi}_n \cdot \mathbf{P}^{(0)} \cdot \mathbf{P}^{(1)} \cdots \mathbf{P}^{(r-1)} & \text{non-homogeneous} \end{cases}$$

$${}_r p_{ij}^{(n)} = P(X_{n+r} = j | X_n = i) \quad \begin{matrix} {}_r p_{ii}^{*(n)} \leq {}_r p_{ii}^{(n)} \\ {}_r p_{ii}^{*(n)} = {}_r p_{ii}^{(n)} \end{matrix} \quad \text{if cannot reenter state } i \text{ once left } i$$

$$\lambda_{ij}(s) = \text{force of transition.} \quad \lambda_i(s) = \sum_{j=1}^{i-1} \lambda_{ij}(s) + \sum_{j=i+1}^m \lambda_{ij}(s) = \text{force of transitioning out.}$$

$$\frac{d}{dr} ({}_r p_{ij}^{(t)}) = \sum_{k \neq j} \left( {}_r p_{ik}^{(t)} \lambda_{kj}(t+r) - {}_r p_{ij}^{(t)} \cdot \lambda_{jk}(t+r) \right) = \sum_{k \neq j} \left( {}_r p_{ik}^{(t)} \lambda_{kj}(t+r) \right) - {}_r p_{ij}^{(t)} \cdot \lambda_j(t+r)$$

Chapter 12 MQR Multiple Life Functions

The Joint Life and Last Survivor Statuses

$$T_{xy} = \min(T_x, T_y) \quad T_{\overline{xy}} = \max(T_x, T_y) \quad f_{xy}(t) = {}_t p_{xy} \mu_{x+t:y+t} \quad {}_t p_{xy} = S_{xy}(t) = \exp\left(-\int_0^t \mu_{x+r:y+r} dr\right)$$

$$S_{\overline{xy}}(t) = {}_t p_x + {}_t p_y - {}_t p_{xy} \quad f_{\overline{xy}}(t) = {}_t p_x \mu_{x+t} + {}_t p_y \mu_{y+t} - {}_t p_{xy} \mu_{x+t:y+t}$$

Fundamental Symmetric Relations (from  $\min(a, b) + \max(a, b) = a + b$ )

$$\begin{aligned} T_{xy} + T_{\overline{xy}} &= T_x + T_y && \text{(Random Variable)} \\ {}_t p_{xy} + {}_t p_{\overline{xy}} &= {}_t p_x + {}_t p_y && \text{or } S_{xy}(t) + S_{\overline{xy}}(t) = S_x(t) + S_y(t) \quad \text{(Survival Function)} \\ {}_t q_{xy} + {}_t q_{\overline{xy}} &= {}_t q_x + {}_t q_y && \text{or } F_{xy}(t) + F_{\overline{xy}}(t) = F_x(t) + F_y(t) \quad \text{(Distribution Function)} \\ f_{xy}(t) + f_{\overline{xy}}(t) &= f_x(t) + f_y(t) && \text{(Density Function)} \end{aligned}$$

regardless of whether  $T_x$  and  $T_y$  are independent.

Deferred probability for last survivor:  $P(m \leq K_{\overline{xy}}^* < m+n) = {}_m | n q_{\overline{xy}} = {}_{m+n} q_{\overline{xy}} - {}_m q_{\overline{xy}} = {}_m | n q_x + {}_m | n q_y - {}_m | n q_{xy}$

Two Independent Lifetimes

$$\begin{aligned} \mu_{x+t:y+t} &= \mu_{x+t} + \mu_{y+t} \\ {}_t p_{xy} = \Pr(T_x > t \text{ and } T_y > t) &= {}_t p_x {}_t p_y && {}_t q_{\overline{xy}} = \Pr(T_x \leq t \text{ and } T_y \leq t) = {}_t q_x {}_t q_y \end{aligned}$$

Force of Mortality of the Last Survivor Status:  $\mu_{x+t:y+t} = \frac{{}_t p_x \mu_{x+t} + {}_t p_y \mu_{y+t} - {}_t p_{xy} \mu_{x+t:y+t}}{{}_t p_{\overline{xy}}}$

Mean, Variance and Covariance of Two Lifetimes

$$\begin{aligned} \overset{\circ}{e}_{xy} &= E(T_{xy}) = \int_0^\infty t \cdot f_{xy}(t) dt = \int_0^\infty {}_t p_{xy} dt && e_{xy} = E(K_{xy}^*) = \sum_{k=1}^\infty k p_{xy} \\ \overset{\circ}{e}_{\overline{xy}} &= E(T_{\overline{xy}}) = \int_0^\infty t \cdot f_{\overline{xy}}(t) dt = \int_0^\infty t \cdot ({}_t p_x \mu_{x+t} + {}_t p_y \mu_{y+t} - {}_t p_{xy} \mu_{x+t:y+t}) dt = \overset{\circ}{e}_x + \overset{\circ}{e}_y - \overset{\circ}{e}_{xy} \\ e_{\overline{xy}} &= E(K_{\overline{xy}}^*) = \sum_{k=1}^\infty k p_{\overline{xy}} = e_x + e_y - e_{xy} && E(T_{xy}^2) = \int_0^\infty t^2 f_{xy}(t) dt = 2 \int_0^\infty t \cdot {}_t p_{xy} dt && E(T_{\overline{xy}}^2) = 2 \int_0^\infty t \cdot {}_t p_{\overline{xy}} dt \end{aligned}$$

$$E(T_x \cdot T_y) = \int_0^\infty \int_0^\infty t_x \cdot t_y \cdot f(t_x, t_y) dt_x dt_y \quad Cov(T_x, T_y) = E(T_x \cdot T_y) - E(T_x) \cdot E(T_y)$$

$$E(T_{xy}) + E(T_{\overline{xy}}) = E(T_x) + E(T_y)$$

$$Var(T_{xy}) + Var(T_{\overline{xy}}) = Var(T_x) + Var(T_y) - 2[(E(T_x) - E(T_{xy})) (E(T_y) - E(T_{xy}))]$$

$$Cov(T_{xy}, T_{\overline{xy}}) = Cov(T_x, T_y) + [E(T_x) - E(T_{xy})] \times [E(T_y) - E(T_{xy})] \quad \text{if } T_x \text{ \& } T_y \text{ independent} \quad \left( \overset{\circ}{e}_x - \overset{\circ}{e}_{xy} \right) \left( \overset{\circ}{e}_y - \overset{\circ}{e}_{xy} \right)$$

Statuses Involving the Order of Death: Contingent Probabilities for Independent Lives

$${}_t q_{1_{xy}} = \int_0^t \Pr(T_y > T_x | T_x = u) f_x(u) du = \int_0^t {}_u p_y {}_u p_x \mu_{x+u} du,$$

$${}_t q_{2_{xy}} = \int_0^t \Pr(T_x < T_y | T_y = u) f_y(u) du = \int_0^t {}_u q_x {}_u p_y \mu_{y+u} du \quad {}_t q_{1_{xy}} + {}_t q_{\overline{1}_{xy}} = {}_t q_{xy} \quad {}_t q_{2_{xy}} + {}_t q_{\overline{2}_{xy}} = {}_t q_{\overline{xy}}$$

Symmetric Relation between Joint and Last Survivor Continuous Insurance

$$\overline{A}_{xy} + \overline{A}_{\overline{xy}} = \overline{A}_x + \overline{A}_y$$

similar relations hold for  $n$ -year term, pure endowment, and endowment insurances.

Covariance between Joint and Last Survivor Benefits

$$Cov(v^{T_{xy}}, v^{T_{\overline{xy}}}) = Cov(v^{T_x}, v^{T_y}) + (\overline{A}_x - \overline{A}_{xy}) (\overline{A}_y - \overline{A}_{xy})$$

Similar relations hold for  $n$ -year term, pure endowment, and endowment insurances.

1. Relation between Insurances and Annuities

$$\overline{a}_{xy} = \frac{1 - \overline{A}_{xy}}{\delta}, \quad \overline{a}_{\overline{xy}} = \frac{1 - \overline{A}_{\overline{xy}}}{\delta} \quad \ddot{a}_{xy} = \frac{1 - A_{xy}}{d}, \quad \ddot{a}_{\overline{xy}} = \frac{1 - A_{\overline{xy}}}{d}$$

Similar relations hold for  $n$ -year endowment insurances and annuities.

2. Fully Discrete Insurances and Annuities

$$\begin{aligned} \ddot{a}_{xy} &= \sum_{k=0}^{\infty} v^k \cdot {}_k p_{xy} & A_{xy} &= E \left[ v^{K_{xy}^*} \right] = \sum_{k=1}^{\infty} v^k \cdot {}_{k-1} | q_{xy} \\ \ddot{a}_{\overline{xy}} &= \sum_{k=0}^{\infty} v^k \cdot {}_k p_{\overline{xy}} & A_{\overline{xy}} &= E \left[ v^{K_{\overline{xy}}^*} \right] = \sum_{k=1}^{\infty} v^k \cdot {}_{k-1} | q_{\overline{xy}} = A_x + A_y - A_{xy} \end{aligned}$$

### 3. Reversionary Annuities (payment only when one life fails until the other also fails)

Payment to (y) when (x) has failed:  $a_{x|y} = \sum_{k=1}^{\infty} v^k ({}_k q_x \cdot {}_k p_y) = \sum_{k=1}^{\infty} v^k ({}_k p_y - {}_k p_{xy}) = a_y - a_{xy}$

n-yrs (at most) pmt to (x) when (y) has failed:  $a_{y|x:n} = \sum_{k=1}^n v^k ({}_k q_x \cdot {}_k p_y) = \sum_{k=1}^n v^k ({}_k p_y - {}_k p_{xy}) = a_{y:n} - a_{xy:n}$

Continuous Payment to (y) when (x) has failed:  $\bar{a}_{x|y} = \int_0^{\infty} v^t ({}_t q_x \cdot {}_t p_y) dt = \int_0^{\infty} v^t ({}_t p_y - {}_t p_{xy}) dt = \bar{a}_y - \bar{a}_{xy}$

$$P(a_{y|x}) = \frac{a_{y|x}}{\ddot{a}_{xy}} = \frac{a_y - a_{xy}}{\ddot{a}_{xy}}. \quad {}_t V(a_{y|x}) = \begin{cases} a_{y+t|x+t} - P(a_{y|x}) \cdot \ddot{a}_{x+t:y+t} & \text{both survives} \\ a_{x+t} & \text{if (x) survives and (y) fails} \\ 0 \text{ since contract expired} & \text{if (x) fails and (y) survives} \end{cases}$$

$$a_{\overline{xy}} = a_{x|y} + a_{y|x} + a_{xy}$$

### 4. Contingent Insurance

$$\bar{A}_{1|xy} = \int_0^{\infty} v^t \cdot {}_t p_{xy} \mu_{x+t} dt \quad \bar{A}_{2|xy} = \int_0^{\infty} v^t \cdot {}_t p_x \mu_{x+t} (1 - {}_t p_y) dt = \bar{A}_x - \bar{A}_{1|xy}$$

$$\bar{A}_{1|xy} + \bar{A}_{1|xy} = \bar{A}_{xy} \quad \bar{A}_{2|xy} + \bar{A}_{1|xy} = \bar{A}_{\overline{xy}}$$

### 5. m-thly payable multiple life benefits

under UDD:  $\ddot{a}_{xy}^{(m)} \approx \alpha(m) \ddot{a}_{xy} - \beta(m) \quad \ddot{a}_{\overline{xy}}^{(m)} \approx \alpha(m) \ddot{a}_{\overline{xy}} - \beta(m) \quad A_{xy}^{(m)} \approx \frac{i}{i^{(m)}} A_{xy} \quad A_{\overline{xy}}^{(m)} \approx \frac{i}{i^{(m)}} A_{\overline{xy}}$

$$\alpha(m) = s_{1|}^{(m)} \ddot{a}_{1|}^{(m)} = \frac{id}{i^{(m)} d^{(m)}} \quad \beta(m) = \frac{s_{1|}^{(m)} - 1}{d^{(m)}} = \frac{i - i^{(m)}}{i^{(m)} d^{(m)}}$$

non-UDD (Woolhouse formula):  $\ddot{a}_{xy}^{(m)} \approx \ddot{a}_{xy} - \frac{m-1}{2m} - \frac{m^2-1}{12m^2} (\delta + \mu_{xy})$

$$\lim_{m \rightarrow \infty} \ddot{a}_{xy}^{(m)} = \bar{a}_{xy} \approx \bar{a}_{xy} - \frac{1}{2} - \frac{1}{12} (\delta + \mu_{xy})$$

### Premiums and Reserves

$$P_{xy} = \frac{A_{xy}}{\ddot{a}_{xy}} \quad P_{\overline{xy}} = \frac{A_{\overline{xy}}}{\ddot{a}_{\overline{xy}}} \quad {}_t V_{xy} = A_{x+t:y+t} - P_{xy} \cdot \ddot{a}_{x+t:y+t}$$

$${}_t V_{\overline{xy}} = \begin{cases} A_{x+t:y+t} - P_{\overline{xy}} \cdot \ddot{a}_{x+t:y+t} & \text{if (x) and (y) survives} \\ A_{x+t} - P_{\overline{xy}} \cdot \ddot{a}_{x+t} & \text{if (x) survives and (y) fails} \\ A_{y+t} - P_{\overline{xy}} \cdot \ddot{a}_{y+t} & \text{if (x) fails and (y) survives} \end{cases}$$

### Dependent Life Models - Common Shock Model

$$\mu_{x+t} = \mu_{x+t}^* + \mu_t^c \stackrel{\text{if constant}}{=} \mu_{x+t}^* + \lambda \quad \mu_{y+t} = \mu_{y+t}^* + \mu_t^c \stackrel{\text{if constant}}{=} \mu_{y+t}^* + \lambda$$

$$\mu_{x+t:y+t} = \mu_{x+t}^* + \mu_{y+t}^* + \lambda \quad {}_t p_{xy} = \exp\left(-\int_0^t [\mu_{x+t}^* + \mu_{y+t}^* + \lambda] dt\right) = {}_t p_x^* \cdot {}_t p_y^* \cdot e^{-\lambda t}$$

$${}_t p_x = {}_t p_x^* \cdot e^{-\lambda t} \quad {}_t p_y = {}_t p_y^* \cdot e^{-\lambda t} \quad \text{Note that } \mu_{xy+t} \neq \mu_{x+t} + \mu_{y+t} \quad \text{and} \quad {}_t p_{xy} \neq {}_t p_x \cdot {}_t p_y$$

### An Exponential Common Shock Model with Constant Force of Transitions (From ACTEX MLC manual)

$$T_x \sim \text{Exp}(\mu_x^* + \lambda), \quad T_y \sim \text{Exp}(\mu_y^* + \lambda), \quad T_{xy} \sim \text{Exp}(\mu_x^* + \mu_y^* + \lambda)$$

$$\bar{A}_x = \frac{\mu_x^* + \lambda}{\mu_x^* + \lambda + \delta} \quad \bar{A}_{xy} = \frac{\mu_x^* + \mu_y^* + \lambda}{\mu_x^* + \mu_y^* + \lambda + \delta} \quad \bar{a}_{xy} = \frac{1}{\mu_x^* + \mu_y^* + \lambda + \delta} \quad \bar{A}_{\overline{xy}} = \bar{A}_x + \bar{A}_y - \bar{A}_{xy}$$

## Chapter 13 MQR Multiple Decrement Models: Theory

**OBJECTIVES:** 1.To understand the concept of a multiple decrement table

2.To understand the force of decrement

3.To construct a multiple decrement model using associated single decrements and to apply various assumptions to calculate rates for discrete jumps.

### 13.1 Discrete Multiple Decrement Models

$$q_x^{(\tau)} = q_x^{(1)} + q_x^{(2)} + \dots + q_x^{(m)} = \sum_{j=1}^m q_x^{(j)} \quad (13.1) \quad p_x^{(\tau)} = 1 - q_x^{(\tau)} \quad (13.2)$$

$${}_n p_x^{(\tau)} = 1 - {}_n q_x^{(\tau)} \quad (13.7e) \quad \ell_{x+n}^{(\tau)} = \ell_x^{(\tau)} \cdot {}_n p_x^{(\tau)} \quad (13.7f)$$

$$\ell_x^{(\tau)} = \sum_{j=1}^m \ell_x^{(j)} \quad (13.6) \quad d_x^{(j)} = \ell_x^{(\tau)} \cdot q_x^{(j)} \quad (13.7a)$$

$$d_x^{(\tau)} = \sum_{j=1}^m d_x^{(j)} = \ell_x^{(\tau)} \cdot q_x^{(\tau)} \quad (13.3 \& 13.7b) \quad {}_n d_x^{(j)} = \sum_{t=0}^{n-1} d_{x+t}^{(j)} = \ell_x^{(\tau)} \cdot {}_n q_x^{(j)} \quad (13.4 \& 13.7c)$$

$${}_n d_x^{(\tau)} = \sum_{j=1}^m {}_n d_x^{(j)} = \ell_x^{(\tau)} \cdot {}_n q_x^{(\tau)} \quad (13.5a \text{ \& } 13.7d) \quad {}_n q_x^{(\tau)} = {}_n d_x^{(\tau)} / \ell_x^{(\tau)} = \sum_{j=1}^m {}_n q_x^{(j)} \quad (13.5b)$$

### 13.1.2 Random Variable Analysis

The *joint probability function* of  $K_x^*$  and  $J_x$  is  $\Pr(K_x^* = k \cap J_x = j) = {}_{k-1} q_x^{(j)} = \frac{d_{x+k-1}^{(j)}}{\ell_x^{(\tau)}}$  (13.8)

The *marginal probability functions* are

i)  $\Pr(K_x^* = k) = \sum_{j=1}^m \Pr(K_x^* = k \cap J_x = j) = {}_{k-1} q_x^{(\tau)} = \frac{d_{x+k-1}^{(1)} + \dots + d_{x+k-1}^{(m)}}{\ell_x^{(\tau)}} = \sum_{j=1}^m \frac{d_{x+k-1}^{(j)}}{\ell_x^{(\tau)}}$  (13.9)

ii)  $\Pr(J_x = j) = \sum_{k=1}^{\infty} \Pr(K_x^* = k \cap J_x = j) = \sum_{k=1}^{\infty} \frac{d_{x+k-1}^{(j)}}{\ell_x^{(\tau)}}$  (13.10)

### 13.2 Theory of Competing Risks

#### 13.3 Continuous Multiple Decrement Models

$$\mu_{x+t}^{(j)} = \frac{-\frac{d}{dt} {}_t p_x^{(j)}}{{}_t p_x^{(j)}} \quad (13.12a) \quad \mu_{x+t}^{(\tau)} = \frac{-\frac{d}{dt} {}_t p_x^{(\tau)}}{{}_t p_x^{(\tau)}} \quad (13.13a) \quad {}_t p_x^{(j)} = \exp\left(-\int_0^t \mu_{x+s}^{(j)} ds\right) \quad (13.12b)$$

$${}_t q_x^{(j)} = \int_0^t {}_s p_x^{(j)} \mu_{x+s}^{(j)} ds = 1 - {}_t p_x^{(j)} \quad \text{Survival Probability } {}_t p_x^{(\tau)} = \exp\left(-\int_0^t \mu_{x+s}^{(\tau)} ds\right) \quad (13.13b)$$

$$f_{x^{(j)}}(t) = {}_t p_x^{(j)} \cdot \mu_{x+t}^{(j)} \quad (13.14) \quad F_{x^{(j)}}(t) = \Pr[T_x^{(j)} \leq t] = \int_0^t f_{x^{(j)}}(s) ds = \int_0^t {}_s p_x^{(j)} \cdot \mu_{x+s}^{(j)} ds \quad (13.15)$$

$$\begin{aligned} \mu_{x+t}^{(\tau)} &= \frac{-\frac{d}{dt} {}_t p_x^{(\tau)}}{{}_t p_x^{(\tau)}} = -\frac{d}{dt} \ln {}_t p_x^{(\tau)} = -\frac{d}{dt} \ln [{}_t p_x^{(1)} \cdot {}_t p_x^{(2)} \cdot \dots \cdot {}_t p_x^{(m)}] \\ &= \left(-\frac{d}{dt} \ln {}_t p_x^{(1)}\right) + \left(-\frac{d}{dt} \ln {}_t p_x^{(2)}\right) + \dots + \left(-\frac{d}{dt} \ln {}_t p_x^{(m)}\right) = \mu_{x+t}^{(1)} + \mu_{x+t}^{(2)} + \dots + \mu_{x+t}^{(m)} = \sum_{j=1}^m \mu_{x+t}^{(j)} \end{aligned} \quad (13.17)$$

*Fundamental Relation Between Primed and Unprimed Rates:*  ${}_t p_x^{(\tau)} = \exp\left(-\sum_{j=1}^m \int_0^t \mu_{x+s}^{(j)} ds\right) = \prod_{j=1}^m {}_t p_x^{(j)}$  (13.16)

$${}_t q_x^{(j)} = \int_0^t f_{T,J}(s, j) ds = \int_0^t {}_s p_x^{(\tau)} \cdot \mu_{x+s}^{(j)} ds \quad (13.18) \quad {}_t q_x^{(\tau)} = \int_0^t {}_s p_x^{(\tau)} \cdot \mu_{x+s}^{(\tau)} ds \quad (13.20)$$

$$\frac{d}{dt} {}_t q_x^{(j)} = \frac{d}{dt} \int_0^t {}_s p_x^{(\tau)} \cdot \mu_{x+s}^{(j)} ds = {}_t p_x^{(\tau)} \cdot \mu_{x+t}^{(j)} \rightarrow \mu_{x+t}^{(j)} = \frac{\frac{d}{dt} {}_t q_x^{(j)}}{{}_t p_x^{(\tau)}} \quad (13.19)$$

**Joint Distribution of  $T_x$  and  $J_x$**   $\Pr(t < T_x \leq t + dt \text{ and } J_x = j) \approx {}_t p_x^{(\tau)} \mu_{x+t}^{(j)} dt, \quad {}_t q_x^{(j)} = \int_0^t {}_s p_x^{(\tau)} \mu_{x+s}^{(j)} ds.$

#### 13.4.1 Uniform Distribution of Decrements in the Multiple Decrement Context

$${}_t q_x^{(j)} = t \cdot q_x^{(j)} \quad (13.21) \quad q_x^{(j)} = {}_t p_x^{(\tau)} \cdot \mu_{x+t}^{(j)} \quad (13.22)$$

$${}_t q_x^{(\tau)} = t \cdot q_x^{(\tau)} \quad (13.23) \quad {}_t p_x^{(\tau)} = 1 - t \cdot q_x^{(\tau)} \quad (13.24)$$

$$\mu_{x+t}^{(j)} = \frac{q_x^{(j)}}{{}_t p_x^{(\tau)}} = \frac{q_x^{(j)}}{1 - t \cdot q_x^{(\tau)}} \quad (13.25) \quad {}_t p_x^{(j)} = \exp\left[\frac{q_x^{(j)}}{q_x^{(\tau)}} \cdot \ln(1 - t \cdot q_x^{(\tau)})\right] = (1 - t \cdot q_x^{(\tau)})^{q_x^{(j)}/q_x^{(\tau)}} \quad (13.26)$$

#### 13.4.2 Uniform Distribution in the Associated Single-Decrement Tables

$${}_t q_x^{(j)} = t \cdot q_x^{(j)} \quad (13.27) \quad {}_t p_x^{(j)} \cdot \mu_{x+t}^{(j)} = q_x^{(j)} \quad (13.28)$$

Double decrement case:  $q_x^{(1)} = \int_0^1 (1 - t \cdot q_x^{(2)}) \cdot q_x^{(1)} dt = q_x^{(1)} \left(1 - \frac{1}{2} \cdot q_x^{(2)}\right)$  (13.29a)

$$q_x^{(2)} = q_x^{(2)} \left(1 - \frac{1}{2} \cdot q_x^{(1)}\right) \quad (13.29b)$$

Triple Decrement case:  $q_x^{(1)} = q_x^{(1)} \left[1 - \frac{1}{2} (q_x^{(2)} + q_x^{(3)}) + \frac{1}{3} (q_x^{(2)} \cdot q_x^{(3)})\right]$  (13.30)

#### Miscellaneous Results (From ACTEX MLC manual)

1. Assumptions on the single decrement table.

##### Backing out the Unprimed Rates from Primed Rates

$${}_s q_x^{(i)} = \int_0^s {}_t p_x^{(\tau)} \mu_{x+t}^{(i)} dt = \int_0^s \left[ \prod_{j=1, j \neq i}^m {}_t p_x^{(j)} \right] {}_t p_x^{(i)} \mu_{x+t}^{(i)} dt$$

2. **Constant Force Assumption for Multiple Decrements**

For any  $t \in [0, 1]$  and integer-valued  $x$ ,

(i)  ${}_t p_x^{(\tau)} = \left[ p_x^{(\tau)} \right]^t$  (survival probability for **fractional** ages)

(ii) *Ratio Property* :  $\frac{{}_t q_x^{(i)}}{{}_t q_x^{(\tau)}} = \frac{\mu_{x+s}^{(i)}}{\mu_{x+s}^{(\tau)}}$  for any  $s \in [0, 1]$  (To get unprimed rates from (i) )

(iii) *Partition Property* :  ${}_t p_x^{t(i)} = \left[ {}_t p_x^{(\tau)} \right]^{q_x^{(i)}/q_x^{(\tau)}}$  (To get primed rates from unprimed rates from (i) )

### 3. Uniform Distribution of Death (UDD) for Multiple Decrement (MUDD) Table

For any  $t \in [0, 1]$  and integer-valued  $x$ ,

(i)  ${}_t p_x^{(\tau)} \mu_{x+t}^{(i)} = q_x^{(i)}$  or equivalently  $\mu_{x+t}^{(i)} = \frac{q_x^{(i)}}{1 - t q_x^{(\tau)}}$  for  $t \neq 1$

(ii) *Ratio Property* :  $\frac{{}_t q_x^{(i)}}{{}_t q_x^{(\tau)}} = \frac{\mu_{x+s}^{(i)}}{\mu_{x+s}^{(\tau)}}$  for any  $s \in [0, 1]$

(iii) *Partition Property* :  ${}_t p_x^{t(i)} = \left[ {}_t p_x^{(\tau)} \right]^{q_x^{(i)}/q_x^{(\tau)}}$  (To get primed rates from unprimed rates  ${}_t q_x^{(i)}$  and  ${}_t p_x^{(\tau)}$ )

**Discrete jumps:** Handling Both Discrete and Continuous Decrement

1)  ${}_s q_x^{(i)} = \int_0^s \left[ \prod_{j=1, j \neq i}^m {}_t p_x^{t(j)} \right] {}_t p_x^{t(i)} \mu_{x+t}^{(i)} dt$  holds when decrement  $i$  is **continuous**.

2)  ${}_s q_x^{(i)} = \sum_{t_k \leq s} \left[ \prod_{j=1, j \neq i}^m {}_{t_k} p_x^{t_k(j)} \right] \Delta(t_k q_x^{t_k(i)})$  holds when decrement  $i$  is **discrete**

Here  $t_k$  are the jump times and  $\Delta(t_k q_x^{t_k(i)})$  is the jump size at time  $t_k$ .

## Chapter 14 MQR Multiple Decrement Models: (Applications)

### 14.1 Actuarial Present Value

$$A_x = \sum_{k=1}^{\infty} v^k \cdot \Pr(K_x^* = k) \quad (14.1) \quad A_x^{(j)} = \sum_{k=1}^{\infty} v^k \cdot \Pr(K_x^* = k \cap J_x = j) \quad (14.2)$$

If the **time** and **cause** of decrement are **independent**,

$$A_x^{(j)} = \sum_{k=1}^{\infty} v^k \cdot \Pr(K_x^* = k) \cdot \Pr(J_x = j) \quad (14.3a) \quad \text{or} \quad A_x^{(j)} = \sum_{k=1}^{\infty} v^k \cdot {}_{k-1} p_x^{(\tau)} \cdot q_{x+k-1}^{(j)} \quad (14.3b)$$

For benefit paid at the **instant of failure**  $\bar{A}_x^{(j)} = \int_0^{\infty} v^t \cdot {}_t p_x^{(\tau)} \cdot \mu_{x+t}^{(j)} dt$  (14.4)

**14.2 Asset Shares**  $[_0 AS + G(1 - r_1) - e_1](1 + i) = b_1^{(1)} \cdot q_x^{(1)} + b_1^{(2)} \cdot q_x^{(2)} + {}_1 AS \cdot p_x^{(\tau)}$ , (14.5a)

$$\text{so } {}_1 AS = \frac{[_0 AS + G(1 - r_1) - e_1](1 + i) - b_1^{(1)} \cdot q_x^{(1)} - b_1^{(2)} \cdot q_x^{(2)}}{p_x^{(\tau)}}. \quad (14.5b)$$

In general,  $[_{k-1} AS + G(1 - r_k) - e_k](1 + i) = b_k^{(1)} \cdot q_{x+k-1}^{(1)} + b_k^{(2)} \cdot q_{x+k-1}^{(2)} + {}_k AS \cdot p_{x+k-1}^{(\tau)}$ , (14.6a)

$$\text{so } {}_k AS = \frac{[_{k-1} AS + G(1 - r_k) - e_k](1 + i) - b_k^{(1)} \cdot q_{x+k-1}^{(1)} - b_k^{(2)} \cdot q_{x+k-1}^{(2)}}{p_{x+k-1}^{(\tau)}}. \quad (14.6b) \quad U_k = {}_k AS - {}_k V^G. \quad (14.7)$$

### 14.3 Non-Forfeiture Options

14.3.1 Cash Value  ${}_t CV_x$

14.3.2 Reduced Paid-up Insurance

$$RPU = \frac{{}_t CV_x}{A_{x+t}}, \quad (14.8) \quad {}_t W_x = \frac{{}_t V_x}{A_{x+t}}, \quad (14.9)$$

14.3.3 Extended Term Insurance  ${}_t CV_x = A_{x+t:n}^1$  (14.10)  ${}_t CV_{x:n} = A_{x+t:n-t}^1 + PE \cdot {}_{n-t} E_{x+t}$ , (14.11)

### 14.4 Multi State Model Representation

14.4.2 The Total and Permanent Disability Model

$${}^h \bar{A}_x^{(f)} = \int_0^{\infty} v^r \cdot {}_r p_x^{(\tau)} \cdot \mu_{x+r}^{(j)} dr \quad (14.12a) \quad {}^h \bar{A}_x^{(f)} = \int_0^{\infty} v^r \cdot {}_r p_{11}^{(0)} \cdot \lambda_{13}(r) dr \quad (14.12b)$$

$${}^d \bar{A}_{x+r} = \int_0^{\infty} v^s \cdot {}_s p_{x+r}^d \cdot \mu_{x+r+s}^d ds \quad (14.13a) \quad {}^d \bar{A}_{x+r} = \int_0^{\infty} v^s \cdot {}_s p_{22}^{(r)} \cdot \lambda_{23}(s) ds \quad (14.13b)$$

$${}^h \bar{A}_x^d = \int_0^{\infty} v^r \cdot {}_r p_x^{(\tau)} \cdot \mu_{x+r}^{(d)} \cdot {}^d \bar{A}_{x+r} dr = \int_0^{\infty} v^r \cdot {}_r p_x^{(\tau)} \cdot \mu_{x+r}^{(d)} \left( \int_0^{\infty} v^s \cdot {}_s p_{x+r}^d \cdot \mu_{x+r+s}^d ds \right) dr \quad (14.14a)$$

$${}^h \bar{A}_x^d = \int_0^{\infty} v^r \cdot {}_r p_{11}^{(0)} \cdot \lambda_{12}(r) \left( \int_0^{\infty} v^s \cdot {}_s p_{22}^{(r)} \cdot \lambda_{23}(s) ds \right) dr, \quad (14.14b)$$

$${}^h \bar{a}_x^d = \int_0^{\infty} v^r \cdot {}_r p_x^{(\tau)} \cdot \mu_{x+r}^{(d)} \cdot {}^d \bar{a}_{x+r} dr = \int_0^{\infty} v^r \cdot {}_r p_x^{(\tau)} \cdot \mu_{x+r}^{(d)} \left( \int_0^{\infty} v^s \cdot {}_s p_{x+r}^d ds \right) dr \quad (14.15a)$$

$${}^h \bar{a}_x^d = \int_0^{\infty} v^r \cdot {}_r p_{11}^{(0)} \cdot \lambda_{12}(r) \left( \int_0^{\infty} v^s \cdot {}_s p_{22}^{(r)} ds \right) dr, \quad (14.15b)$$

$$14.4.3 \text{ Disability Model Allowing For Recovery } f(x + \Delta x) \approx f(x) + f'(x) \cdot \Delta x \quad (14.16)$$

$$\begin{aligned} \frac{d}{dr} r p_{11}^{(t)} &= r p_{12}^{(t)} \cdot \lambda_{12}(t+r) - r p_{11}^{(t)} \cdot \lambda_{12}(t+r) && (\text{at } k=2) \\ &+ r p_{13}^{(t)} \cdot \lambda_{31}(t+r) - r p_{11}^{(t)} \cdot \lambda_{13}(t+r) && (\text{at } k \text{ of } 3) \\ &= r p_{12}^{(t)} \cdot \lambda_{12}(t+r) - r p_{11}^{(t)} \cdot [\lambda_{12}(t+r) + \lambda_{13}(t+r)], && (14.17) \end{aligned}$$

$$\begin{aligned} \frac{d}{dr} r p_{12}^{(t)} &= r p_{11}^{(t)} \cdot \lambda_{12}(t+r) - r p_{12}^{(t)} \cdot \lambda_{21}(t+r) && (\text{at } k=1) \\ &+ r p_{13}^{(t)} \cdot \lambda_{32}(t+r) - r p_{12}^{(t)} \cdot \lambda_{23}(t+r) && (\text{at } k \text{ of } 3) \\ &= r p_{11}^{(t)} \cdot \lambda_{12}(t+r) - r p_{12}^{(t)} \cdot [\lambda_{21}(t+r) + \lambda_{23}(t+r)] && (14.18) \end{aligned}$$

$$r + \Delta r p_{ij}^{(0)} \approx r p_{ij}^{(0)} + \frac{d}{dr} r p_{ij}^{(0)} \cdot \Delta r \quad (14.19)$$

$$r + \Delta r p_{11}^{(0)} \approx r p_{11}^{(0)} + \Delta r \left\{ r p_{12}^{(0)} \cdot \lambda_{21}(r) - r p_{11}^{(0)} \cdot [\lambda_{12}(r) + \lambda_{13}(r)] \right\}, \quad (14.20)$$

$$r + \Delta r p_{12}^{(0)} \approx r p_{12}^{(0)} + \Delta r \left\{ r p_{11}^{(0)} \cdot \lambda_{12}(r) - r p_{12}^{(0)} \cdot [\lambda_{21}(r) + \lambda_{23}(r)] \right\}, \quad (14.21)$$

#### 14.4.5 Thiele's Differential Equation in the Multiple Decrement Case

$$\bar{a}_x^{(\tau)} = \int_0^\infty v^r \cdot r p_x^{(\tau)} dr \quad (14.22) \quad \bar{P} = \frac{APVB}{\bar{a}_x^{(\tau)}} = \frac{APV_x^T}{\bar{a}_x^{(\tau)}} = \frac{\sum_{j=1}^m APV_x^{(j)}}{\bar{a}_x^{(\tau)}} \quad (14.23)$$

$${}^d \bar{a}_{x+t} = \int_0^\infty v^s \cdot {}_s p_{x+t}^d dr \quad (14.24) \quad \frac{d}{dt} {}^h \bar{V} = \bar{P} + \delta \cdot {}^h \bar{V} - \mu_{x+t}^{(f)} (1 - {}^h \bar{V}) - \mu_{x+t}^{(d)} ({}^d \bar{V} - {}^h \bar{V}). \quad (14.25)$$

$$\frac{d}{dt} {}^d \bar{V} = \delta \cdot {}^d \bar{V} - 1 - {}^d \mu_{x+t}^{(f)} (1 - {}^d \bar{V}) - {}^d \mu_{x+t}^{(r)} ({}^h \bar{V} - {}^d \bar{V}) \quad (14.26)$$

$$\frac{{}^h \bar{V} - {}^d \bar{V}}{\Delta t} \approx \bar{P} + \delta \cdot {}^h \bar{V} - \mu_{x+t}^{(f)} (1 - {}^h \bar{V}) - \mu_{x+t}^d ({}^d \bar{V} - {}^h \bar{V}),$$

$$\text{or } {}^h \bar{V} - {}^d \bar{V} \approx \Delta t \left\{ \bar{P} + \delta \cdot {}^h \bar{V} - \mu_{x+t}^{(f)} (1 - {}^h \bar{V}) - \mu_{x+t}^d ({}^d \bar{V} - {}^h \bar{V}) \right\} \quad (14.27)$$

$${}^d \bar{V} - {}^h \bar{V} \approx \Delta t \left\{ \delta \cdot {}^d \bar{V} - 1 - {}^d \mu_{x+t}^{(f)} (1 - {}^d \bar{V}) - {}^d \mu_{x+t}^{(r)} ({}^h \bar{V} - {}^d \bar{V}) \right\} \quad (14.28)$$

### 14.6 Gain and Loss Analysis

Profit with all anticipated factors:

$$P(0) = [{}_t V + G(1 - r_{t+1}) - e_{t+1}] (1 + i_{t+1}) - \left[ (b_{t+1}^{(1)} + s_{t+1}^{(1)}) \cdot q_{x+t}^{(1)} + (b_{t+1}^{(2)} + s_{t+1}^{(2)}) \cdot q_{x+t}^{(2)} + p_{x+t}^{(\tau)} \cdot {}_{t+1} V \right] \quad (14.39)$$

Profit with some actual experience in place of anticipated factors:

$$P(1) = (14.39) \text{ with all anticipated factors except actual value for 1 factor.}$$

$$P(2) = (14.39) \text{ with all anticipated factors except actual value for 2 factors}$$

$$P(3) = (14.39) \text{ with all anticipated factors except actual value for 3 factors.}$$

$$P(4) = (14.39) \text{ with all anticipated factors except actual value for 4 factors.}$$

$$\text{Gain from factor whose gain is calculated first is } G^{F_1} = P(1) - P(0) \quad (14.40a)$$

$$\text{Gain from factor whose gain is calculated second is } G^{F_2} = P(2) - P(1) \quad (14.40b)$$

$$\text{Gain from factor whose gain is calculated third is } G^{F_3} = P(3) - P(2) \quad (14.40c)$$

$$\text{Gain from factor whose gain is calculated fourth is } G^{F_4} = P(4) - P(3) \quad (14.40d)$$

$$\text{Gain from factor whose gain is calculated } k\text{th is } G^{F_k} = P(k) - P(k-1)$$

$$\text{Total gain } G^T = G^{F_1} + G^{F_2} + G^{F_3} + G^{F_4} = P(4) - P(0) \quad (14.41) \quad \text{Total gain } G^T = \sum_{k=1}^{last} G^{F_k} = P(last) - P(0)$$

When death occurs throughout year but withdrawal only at end of year, the anticipated profit expression is

$$P(0) = [{}_t V + G(1 - r_{t+1}) - e_{t+1}] (1 + i_{t+1}) - \left[ (b_{t+1}^{(1)} + s_{t+1}^{(1)}) \cdot q_{x+t}^{(1)} + (b_{t+1}^{(2)} + s_{t+1}^{(2)}) \left( 1 - q_{x+t}^{(1)} \right) \cdot q_{x+t}^{(2)} + p_{x+t}^{(\tau)} \cdot {}_{t+1} V \right] \quad (14.42)$$

### 14.5 Defined Benefit (DB) Pension Plans

#### 14.5.1 Normal Retirement (NR) Benefits

$$\text{Projected Annual Benefit } PAB_z = 0.01p \cdot YOS_z \cdot FAS_z \quad (14.29)$$

$$\text{Final Annual Salary } FAS_z = \frac{1}{3} \left( \frac{S_{z-3} + S_{z-2} + S_{z-1}}{S_x} \right) \cdot CAS_x \quad (14.30)$$

$$\text{Projected Aggregate Salary } PAS_z = \frac{1}{S_x} \sum_{k=x}^{z-1} S_k \cdot CAS_x \quad (14.31)$$

$$\text{Projected Annual retirement Benefit: } PAB_z = 0.01p \cdot PAS_z \quad (14.32)$$

$$\text{APV of the projected benefit, at age } x: APV_x^{NR} = PAB_z \cdot v^{z-x} \cdot {}_{z-x} p_x^{(\tau)} \cdot r \ddot{a}_z^{(12)}. \quad (14.33)$$

### 14.5.2 Early Retirement (ER) Benefits

$$APV_{35}^{ER} = \sum_{y=60}^{64} PAB_{y+1/2} \cdot \left[ 1 - 0.05 \left( 65 - y - \frac{1}{2} \right) \right] \cdot v^{y+1/2-35} \cdot {}_{y-35}p_{35}^{(\tau)} \cdot q_y^{(\tau)} \cdot r \ddot{a}_{y+1/2}^{(12)} \quad (14.34)$$

### 14.5.3 Withdrawal and other Benefits

Assuming a 5-year vesting rule and assuming employees take their *withdrawal* benefit at NRA, the APV at age 35 is

$$APV_{35}^W = \sum_{y=35+5}^{59} PAB_{y+1/2} \cdot v^{30} \cdot {}_{y-35}p_{35}^{(\tau)} \cdot q_y^{(w)} \cdot {}_{65-y-1/2}p_{y+1/2} \cdot r \ddot{a}_{65}^{(12)} \quad (14.35)$$

### 14.5.4 Funding and Reserving

$$\text{Normal Cost (Early Age)} \quad NC_x^{EAN} = \frac{APV_x^T}{\ddot{a}_{x:z-x}^{(\tau)}} \quad (14.36)$$

$${}_tV_x^T = APV_{x+t}^T - NC_x^{EAN} \cdot \ddot{a}_{x+t:z-x-t}^{(\tau)} \quad (14.37a) \text{ or retrospectively as } {}_tV_x^T = NC_x^{EAN} \cdot \ddot{s}_{x:t}^{(\tau)} \quad (14.37b)$$

$$APV \text{ of the benefit accrued between ages } x \text{ and } x+1: APV_x^{NR} = (AB_{x+1} - AB_x) \cdot v^{z-x} \cdot {}_{z-x}p_x^{(\tau)} \cdot r \ddot{a}_z^{(12)} \quad (14.38)$$

## ACTEX MLC Chapter 9 Study Manual Vol II Multiple Decrement Models: Applications

Thiele's Differential Equation under Multiple Decrement

$$\frac{d}{{dt}} {}_tV^g = G_t(1 - c_t) - e_t + \left( \delta + \mu_{x+t}^{(\tau)} \right) {}_tV^g - \sum_{j=1}^n \left( b_t^{(j)} + E_t^{(j)} \right) \mu_t^{(j)}$$

Recursive Relation for Expected Asset Shares

$$[{}_hAS + G_h(1 - c_h) - e_h](1 + i) = p_{x+h}^{(\tau)} {}_{h+1}AS + q_{x+h}^{(1)} {}_{h+1}CV + q_{x+h}^{(2)} b_{h+1}$$

## ACTEX MLC Chapter 16 Pension Mathematics

### 16.1 Salary Scale Function

$$\frac{s_y}{s_x} = \frac{\text{Salary received in year of age } y \text{ to } y+1}{\text{Salary received in year of age } x \text{ to } x+1} = \frac{\int_y^{y+1} \bar{s}_u du}{\int_x^{x+1} \bar{s}_u du}$$

$$\text{Salary received in year of age } y \text{ to } y+1 = (\text{Salary received in year of age } x \text{ to } x+1) \frac{s_y}{s_x}$$

**Approximations.** idea: Salary rate at age  $x \approx s_{x-0.5}$

$$\text{Salary received in year of age } y \text{ to } y+1 = (\text{Salary received at age } x) \frac{s_y}{s_x} \approx (\text{Salary received at age } x) \frac{s_y}{s_{x-0.5}}$$

### 16.2 Salary Scale Function

Defined Contribution (DC) Plan	future retirement benefit is subject to investment earnings during accumulation phase
Defined Benefit (DB) Plan	future retirement benefit is defined

### 16.3 Setting the DC Contribution Rate

Finding the appropriate contribution rate for a DC plan to meet a target replacement ratio  $R$ , the following are needed

1. target replacement ratio  $R$  and target retirement age  $z$
2. assumptions on (1) investment rate of return, (2) interest rates at retirement, (3) a salary scale and (4) post-retirement mortality model
3. the form the benefits should take

Step 1)	Project <b>future salaries</b> . Use the salary scale.
2)	At retirement date, <b>calculate accumulated value</b> of all contributions. Don't use mortality and withdrawal as DC fund is like a bank account. If a person dies, the accumulated fund is paid out to beneficiaries. If a person withdraws, the accumulated fund is paid to the employee.
3)	Multiply <i>final salary</i> (in step 1) by the <i>replacement ratio</i> to get the annual pension benefits. The calculate the <b>actuarial present value</b> of all pension benefit at retirement.
4)	Solve for <b>contribution rate</b> by equating values in <i>Step 2</i> and <i>Step 3</i> .

$$\text{Handy formula for DC plan: } \sum_{k=1}^n x^k = \frac{x^{n+1} - x}{x - 1}$$

### 16.4 DB Plan and Service Table

Final Salary Plan	$R(x, h, t) = \alpha (AS_{x+h+t-1}) = \alpha \times AS_{x+h-1} \cdot \frac{s_{x+h+t-1}}{s_{x+h-1}}$
Final m-year average salary plan or Final Average Pay (FAP) plan	$R(x, h, t) = \alpha (h+t) \frac{AS_{x+h+t-1} + AS_{x+h+t-2} + \dots + AS_{x+h+t-m}}{s_{x+h+t-1} + s_{x+h+t-2} + \dots + s_{x+h+t-m}}$ $= \alpha (h+t) AS_{x+h-1} \frac{m}{s_{x+h+t-1} + s_{x+h+t-2} + \dots + s_{x+h+t-m}}$ if $\widehat{S}_{FIN} = \frac{AS_{x+h+t-1} + AS_{x+h+t-2} + \dots + AS_{x+h+t-m}}{ms_{x+h-1}}$ then $R(x, h, t) = \alpha (h+t) \widehat{S}_{FIN} = \alpha h \widehat{S}_{FIN} + \alpha t \widehat{S}_{FIN} = \text{accrued} + \text{future service benefit.}$
Career Average Earnings (CAE) plan	$R(x, h, t) = \alpha (h+t) \frac{AS_{x+h+t-1} + AS_{x+h+t-2} + \dots + AS_x}{h+t}$ $= \alpha (h+t) \frac{TPE(x, h, t)}{h+t} = \alpha (h+t) CAE(x, h, t) = \alpha TPE(x, h, t)$ $= \alpha (AS_x + AS_{x+1} + \dots + AS_{x+h-1}) + \alpha (AS_{x+h} + AS_{x+h+1} + \dots + AS_{x+h+t-1})$ $= \alpha TPE(x, h, 0) + \alpha (AS_{x+h} + AS_{x+h+1} + \dots + AS_{x+h+t-1})$ $= \text{accrued benefit} + \text{future service benefit.}$

- 1) Build a model for the time of retirement  $t$  and compute the probability of retiring at age  $x + h + t$  for each  $t$ .
- 2) Compute the APV

Accumulation phase	Multidecrement model with irreversible states: 0 (Active employee), 1 (Withdrawn ( $w$ )), 2 (Permanently Disabled ( $i$ )), 3 (Retired ( $r$ )), and 4 (Died ( $d$ ))
after transition from state 0	3 Single Decrement models
Exact APV	$\int_{nr-x-h}^{\infty} v^t \cdot {}_t p_{x+h}^{(\tau)} l_{x+h+t}^{(r)} R(x, h, t) \cdot r \ddot{a}_{x+h+t}^{(12)} dt$ where $nr$ = normal retirement age.
Approximate APV mult decr table	$\sum_{t=nr-x-h}^{\infty} v^{t+1/2} \cdot {}_t p_{x+h}^{(\tau)} q_{x+h+t}^{(r)} R(x, h, t + \frac{1}{2}) \cdot r \ddot{a}_{x+h+t+1/2}^{(12)}$

**Service Table:** Under total probability law, for any integer age  $x > x_0$ ,  $l_x = l_{x-1} - w_{x-1} - i_{x-1} - r_{x-1} - d_{x-1}$ . (10.2)

$$\begin{aligned} \text{For } x_0 + k \text{ (} k = 0, 1, \dots \text{), } w_{x_0+k} &= l_{x_0} k p_{x_0}^{00} p_{x_0+k}^{01}, & i_{x_0+k} &= l_{x_0} k p_{x_0}^{00} p_{x_0+k}^{02}, \\ r_{x_0+k} &= l_{x_0} k p_{x_0}^{00} p_{x_0+k}^{03}, & d_{x_0+k} &= l_{x_0} k p_{x_0}^{00} p_{x_0+k}^{04}, & l_{x_0+k} &= l_{x_0} k p_{x_0}^{00}. \end{aligned}$$

Pension Plan **Replacement Ratio**,  $R = \frac{\text{pension income in the year after retirement}}{\text{salary in the year before retirement}}$

### 16.5 Funding of DB Plans

PUC	Projected Unit Credit. Uses the average of <b>final</b> $m$ salaries
TUC	Traditional Unit Credit. Use the most <b>recent</b> $m$ salaries.

$${}_t V + C_t = EPV \text{ of benefits of mid-year exits} + v p_x^{00} \cdot {}_{t+1} V \text{ where } C_t = \text{normal cost}$$

$$C_t = EPV \text{ of benefits of mid-year exits} + v p_x^{00} \cdot {}_{t+1} V - {}_t V$$

### Chapter 15 MQR Models with Variable Interest Rates

**15.4 Forward Interest Rates**  $(1 + y_5)^5 = (1 + y_1)^1 \cdot (1 + f_{1,4})^4$ . (15.1)

$$\begin{aligned} (1 + y_4)^4 &= (1 + y_2)^2 \cdot (1 + f_{2,2})^2 & (15.2) & & (1 + y_k)^k \cdot (1 + f_{k,5-k})^{5-k} &= (1 + y_5)^{k+5-k} = (1 + y_5)^5. & (15.3) \\ (1 + y_2)^2 &= (1 + f_{1,1}) (1 + f_{0,1}) \end{aligned}$$

### Chapter 12 ACTEX MLC Study Manual Vol II Interest Rate Risk

**Spot interest rate**  $v(t) = (1 + y_t)^{-t}$       **Forward interest rate**  $(1 + f_{t,k})^k = \frac{(1 + y_{t+k})^{t+k}}{(1 + y_t)^t} = \frac{v(t)}{v(t+k)}$

### ACTEX MLC Chapter 13 Profit Testing

Profit vector  $\text{Pr} = (\text{Pr}_0, \text{Pr}_1, \text{Pr}_2, \text{Pr}_3 \dots)'$   
 Profit **signature**  $\Pi = (\Pi_0, \Pi_1, \Pi_2, \Pi_3, \dots)' = (\text{Pr}_0, \text{Pr}_1, p_x \text{Pr}_2, 2p_x \text{Pr}_3, \dots)'$

Expected profit that emerges in  $(h + 1)^{th}$  year

$$\begin{aligned} \text{Pr}_{h+1} &= N[({}_h V + G_h(1 - c_h) - e_h)(1 + i_{h+1}) - ((b_{h+1} + E_{h+1})q_{x+h} + p_{x+h} \cdot {}_{h+1} V)] \\ &= N[(G_h(1 - c_h) - e_h)(1 + i_{h+1}) - (b_{h+1} + E_{h+1})q_{x+h} + (1 + i_{h+1}){}_h V - p_{x+h} \cdot {}_{h+1} V] \end{aligned} \quad (1)$$

$\text{Pr}_1$  = equation (1) without acquisition cost.       $\text{Pr}_0$  = - all acquisition costs

**Extension to Multiple State Models** (assuming  $N = 1$ )

$$\Pr_{h+1}^{(j)} = [{}_hV^{(j)} + G_h^{(j)}(1 - c_h^{(j)}) - e_h^{(j)}](1 + i_{h+1}) - \sum_{k=0}^n (b_{h+1}^{(jk)} + E_{h+1}^{(jk)} + {}_{h+1}V^{(k)})p_{x+h}^{jk}$$

$$\Pi = (\Pi_0, \Pi_1, \Pi_2, \dots)', \quad \Pi_0 = \Pr_0^{(0)}, \quad \Pi_1 = \sum_{k=0}^n {}_0p_x^{0k} \Pr_t^{(k)} = {}_0p_x^{00} \Pr_1^{(0)} = \Pr_1^{(0)}$$

$$\Pi_t = \sum_{k=0}^n {}_{t-1}p_x^{0k} \Pr_t^{(k)}$$

**Traditional Insurance Policies with Withdrawal** (assuming  $N = 1$ )

$$\Pr_{h+1}^{(0)} = [{}_hV^{(0)} + G_h(1 - c_h) - e_h](1 + i_{h+1}) - [{}_{h+1}V^{(0)}p_{x+h}^{(\tau)} + (b_{h+1} + E_{h+1})q_{x+h}^{(1)} + {}_{h+1}CVq_{x+h}^{(2)}] \quad h \geq 0 \text{ and}$$

$$\Pr_{h+1}^{(1)} = 0, \Pr_{h+1}^{(2)} = 0, h \geq 1 \quad \Pr_0^{(0)} = - \text{acquisition costs} = -\text{pre-contract expenses} - {}_0V$$

$$\Pi_h = {}_{h-1}p_x^{(\tau)} \Pr_h^{(0)}, h \geq 2, \quad \Pi_1 = \Pr_1^{(0)}, \quad \Pi_0 = \Pr_0^{(0)}$$

**Extension to Policies with Continuous Benefit** (assuming  $N = 1$ ): A crude approximation

$$\Pr_{h+1} = [{}_hV^g + G_h(1 - c_h) - e_h](1 + i_{h+1}) - [(1 + i_{h+1})^{1/2}(b_{h+0.5} + E_{h+0.5})q_{x+h} + p_{x+h} \cdot {}_{h+1}V^g].$$

### 13.2 Profit Measures

**1. Net present value (NPV):** 
$$NPV(r) = \sum_{k=0}^n \frac{C_k}{(1+r)^k}.$$

If the  $NPV > 0$ , then the investment is deemed **profitable**.

**2. Internal rate of return (IRR):** The internal rate of return is the zero of the equation

$$NPV(r) = \sum_{k=0}^n \frac{C_k}{(1+r)^k} = 0.$$

### 3. Discounted payback period (DPP)

Given the hurdle rate  $r$ , the **discounted payback period** (also known as the **break-even period**) is the smallest value of  $m$  such that the **partial net present value**  $NPV(r; m) = \sum_{k=0}^m \frac{C_k}{(1+r)^k} \geq 0$ .

DPP is the **time until the investment starts to make a profit**.

### 4. Profit margin

Profit margin is the **NPV of the net cash flows as a percentage of the NPV of the revenues**. Suppose that the revenue cash flows are  $R_0, R_1, R_2, \dots, R_n$ . Then the **profit margin** is 
$$\frac{NPV(r)}{\sum_{k=0}^n \frac{R_k}{(1+r)^k}}$$

For **life insurance**,  $C_k = \Pi_k$ ,  $R_k = G_k \cdot {}_k p_x$ , NPV is expected present value of the profits at issue and the NPV of premiums as revenues (mortality rate taken into account).

$$\text{Profit margin} = \frac{NPV(r)}{\sum_{k=0}^n \frac{G_k \cdot {}_k p_x}{(1+r)^k}} = \frac{\sum_{k=0}^n \frac{\Pi_k}{(1+r)^k}}{\sum_{k=0}^n \frac{G_k \cdot {}_k p_x}{(1+r)^k}}$$

**5. NPV as a proportion of the acquisition costs** 
$$= \frac{1}{\Pi_0} NPV(r) = \frac{1}{\Pi_0} \sum_{k=0}^n \frac{\Pi_k}{(1+r)^k}.$$

### 13.3 Using Profit Test to Compute Premiums and Reserves

**Zeroization** of profit vector

Profit Vector without allowance for reserves	$\Pr = (\Pr_0, \Pr_1, \Pr_2, \Pr_3, \dots)'$
term policy for $t < n - 1$	${}_tV^Z = \max\left(\frac{p_{x+t} \cdot {}_{t+1}V^Z - \Pr_{t+1}}{1 + i_t}, 0\right)$
endowment policy	same as term policy for $t < n - 1$ at $t = n$ , ${}_{n-1}V^Z = \max\left(\frac{-\Pr_n}{1 + i_{n-1}}, 0\right)$
Profit after zeroization	$\Pr_{h+1}^Z = \Pr_{h+1} + (1 + i_{h+1}){}_hV - p_{x+h} \cdot {}_{h+1}V$

**Note:** If the  ${}_{t+1}V^Z = 0$  and  $\Pr_{t+1} > 0$ ,  ${}_tV^Z = \max(-ve, 0) = 0$ .



**Chapter 16 MQR Universal Life Insurance (ULI)**

**16.1 Definition and Basic Mechanics.**

Contract Type	Cost of Insurance ( $COI_t$ ) for ULI	for ULI with Corridor Factor (CF) $COI_t^{CF} = \frac{NAR_t \cdot q_{x+t-1}}{1+i^q}$
A	$\frac{q_{x+t-1}}{1+i^q} (B - (AV_{t-1} + G_t(1-r_t) - e_t)(1+i^c))$ $1 - \left(\frac{q_{x+t-1}}{1+i^q}\right)(1+i^c)$	$\frac{q_{x+t-1}}{1+i^q} (AV_{t-1} + G_t(1-r_t) - e_t)(1+i^c)(f-1)$ $1 + \left(\frac{q_{x+t-1}}{1+i^q}\right)(1+i^c)(f-1)$
B	$\frac{B \cdot q_{x+t-1}}{1+i^q}$	$\frac{q_{x+t-1}}{1+i^q} (AV_{t-1} + G_t(1-r_t) - e_t)(1+i^c)(f-1)$ $1 + \left(\frac{q_{x+t-1}}{1+i^q}\right)(1+i^c)(f-1)$

Death Benefit, Accumulated Value ( $AV_t$ ) and Net Amount at Risk ( $NAR_t$ )

ULI Contract Type	Death Benefit	$AV_t$ for ULI	$NAR_t$ for ULI
A	$B$	$\frac{(AV_{t-1} + G_t(1-r_t) - e_t - B\frac{q_{x+t-1}}{1+i^q})(1+i^c)}{1 - q_{x+t-1}\left(\frac{1+i^c}{1+i^q}\right)}$	$B - AV_t$
B	$B + AV_t$	$(AV_{t-1} + G_t(1-r_t) - e_t - COI_t)(1+i^c)$	$B + AV_t - AV_t = B$
A with Corridor factor	$DB_t = \max(B, f \cdot AV_t)$	$(1+i^c) \times [AV_{t-1} + G_t(1-r_t) - e_t - \max(COI_t^{CF}, COI_t^{wo CF})]$	$DB_t - AV_t = \max[B_t - AV_t, (f-1)AV_t]$
B with Corridor factor	$\max(B + AV_t, f \cdot AV_t)$	same formula as above	$DB_t - AV_t = \max[B_t - AV_t, (f-1)AV_t]$

$i^q$  interest rate in the calculation of premium for death benefit

$i^c$  interest rate in the calculation of saving part of ULI

$f$  corridor factor to use for the minimum death benefit or the minimum insurance amount.

**ULI** **Universal Life Insurance.** Policy is marked by (a) extensive policyholder choice, (b) policyholder participation in interest rate risk, and (c) secondary guarantee features of coverage

**VUL** **Variable Universal Life** insurance. Separate investment accounts for net contributions.

**EIUL** **Equity-Indexed Universal Life** insurance. Interest/investment is credited to contract at rate that depends on some published stock index such as SP500, DJIA, or EAFE index

$SC_t$  **Surrender Charge** at time  $t$ .

$M\&E_t$  **Mortality and Expenses** Charge at time  $t$ .

$NAR_t$  **Net Amount at Risk** at time  $t$ .

$AV_t$  **Account Value** at time  $t$ .

$CV_t$  **Cash Value** at time  $t$ . ( $CV_t = AV_t - SC_t$ )

**NAIC** National Association of Insurance Commissioners

**PG** **Policy Guarantees** (Guarantees given as part of an insurance policy).

**GMP** **Guaranteed Maturity Premium.** Level gross premium sufficient to endow the policy at its maturity date based on the policy guarantees of premium loads, interest rates, and expense and mortality charges.

**GMF** **Guaranteed Maturity Funds.** Calculated based on the roll forward of the GMP and the policy guarantees.

**XXXX16.2 Indexed Universal Life Insurance.**

a) Point-to-point method:  $i_P = \frac{\text{Final Index Closing Value}}{\text{Initial Index Closing Value}} - 1$ , (16.1)

b) Monthly average method:  $i_{MA} = \frac{\frac{1}{12} \sum \text{Monthly Index Closing Values}}{\text{Initial Index Closing Value}} - 1$ . (16.2)

**XXXX16.3 Pricing Considerations**

*Mortality rate, Lapse rate, Expenses, Investment Income.*

Double decrement model:  $p_x^{(\tau)} = 1 - q_x^{(\tau)} = 1 - q_x^{(d)} - q_x^{(w)}$ . (16.3)

Withdrawal at end of year only:  $p_x^{(\tau)} = (1 - q_x^{(d)}) (1 - q_x^{(w)})$ . (16.4)

Pricing for Secondary Guarantees: a) Stipulated premium method, b) Shadow fund method.

**XXXX16.4 Reserving Considerations**

<i>GDB</i>	Guaranteed Death Benefits.
<i>GMB</i>	Guaranteed Maturity Benefits.
<i>PVFB<sub>t</sub></i>	<b>Present Value</b> at time <i>t</i> of the projected <b>Future Benefits</b> .
<i>PVFP<sub>t</sub></i>	<b>Present Value</b> at time <i>t</i> of the <b>Future GMP stream</b> .
<i>CRVM</i>	Commissioner's reserve valuation method
<i>CSV<sub>t</sub></i>	<b>Cash Surrender Value</b> at time <i>t</i> .
<i>AMR</i>	<b>Alternative Minimum Reserves</b> .

Roll Forward =bring a financial value forward to the future .

#### XXXX16.4.1 Basic Universal Life (ULI)

Process for 1983 **NAIC** regulation to define a *minimum reserving standard* for UL products.

a) At policy issue,

1. a guaranteed maturity premium (*GMP*) is calculated as the level gross premium sufficient to endow the policy at its maturity date. The *GMP* is based on the **policy guarantees** of premium loads, interest rates, and expense and mortality charges.

$GMP_0$  is policy guarantees of  $f(\text{premium loads, } i, M\&E)$ .

2. a sequence of **guaranteed maturity funds** (*GMF*) is calculated based on the roll forward of the *GMP* and the policy guarantees

a sequence  $GMF = \text{roll forward of } f(GMP, \text{policy guarantees})$ .

b) At the valuation date, *t*,

3. actual  $AV_t$  determined by the account value roll forward process.

ULI with **variable failure benefit**  $B + AV_t$  (11.25c):  $AV_t = [AV_{t-1} + G_t(1 - r_t) - e_t - v_t q_{x+t-1} \cdot B](1 + i_t)$

ULI with **fixed failure benefit** (11.26b):  $AV_t = [AV_{t-1} + G_t(1 - r_t) - e_t](1 + i_t) - q_{x+t-1} \cdot (B - AV_t)$

$$(11.27) \quad AV_t = \frac{[AV_{t-1} + G_t(1 - r_t) - e_t](1 + i_t) - q_{x+t-1} \cdot B}{p_{x+t-1}}$$

4. the ratio of the actual account value to the *GMF* is calculated as  $r_t = \frac{AV_t}{GMF_t}$ ,  $r_t \leq 1$  (16.5)

5.  $\max(AV_t, GMF_t)$  is projected forward based on the *GMP* and the policy guarantees. This produces a sequence of *GDB* and *GMB*.

6.  $PVFB_t$  and  $PVFP_t$  are calculated using valuation assumptions. Then the pre-floor *CRVM* reserve is defined as

$${}_t V^{pre-floor CRVM} = r_t (PVFB_t - PVFP_t) \quad (16.6) \quad \text{with } r_t \text{ as defined above.}$$

7.  ${}_t V^{floor CRVM} = \max(\frac{1}{2}\text{-month term reserve based on minimum valuation mortality and interest, } CSV_t)$ .

8.  ${}_t V^{final CRVM} = \max({}_t V^{pre-floor CRVM}, {}_t V^{floor CRVM})$ .

The regulation also defines **alternative minimum reserves** (*AMR*).

1. the valuation net premium is calculated at policy issue ( $t = 0$ ) based on the *GMP* and the policy guarantees.

2. If the *GMP* < the valuation net premium (*VNP*),

the reserve held =  $\max(a, b)$

where  $a$  = the reserve calculated using the **actual method** and **assumptions** of the policy + *VNP*,

$b$  = the reserve calculated using the **actual method** but with **minimum valuation** assumptions + *GMP*).

#### XXXX16.4.1 Indexed Universal Life (eIUL)

NAIC Actuarial Guideline 36 (AG 36) specifies the valuation standards for IUL contracts. 3 computational methods:

- 1) The **implied guaranteed rate** (*IGR*) method: which requires insurers to satisfy the hedged-as-required criteria. These criteria set forth a strenuous constraint requiring exact, or nearly exact, hedging, as well as an indexed

interest-crediting term of *not more than one year*.

2) The **CRVM with updated market value** (CRVM/UMV) method:

must be used if the contract has an *indexed interest-crediting term of more than one year*, or if the renewal participation rate guarantee gives an implied guaranteed rate greater than the maximum valuation rate. This method can be volatile when market conditions change.

(3) The **CRVM with updated average market value** (CRVM/UAMV) method:

is a hybrid of the other two, designed for an insurer who **qualifies for the first method** above but **does not wish to satisfy** the *hedged-as-required* criteria.

The CRVM/UMV method has calculation steps as follows:

a) The **issue date** ( $t = 0$ ) calculations are as follows:

1. An **implied guaranteed interest rate** (IGR) for the duration of the initial term,

is the **guaranteed rate** plus the **accumulated option cost** expressed as a percentage of the policy value to which the indexed benefit is applied. In turn, the **accumulated option cost** is the amount needed to provide the index-based benefit in excess of any other interest rate guarantee, accumulated to the end of the initial term at the appropriate maximum valuation rate.

2. An **implied guaranteed rate** for the period **after** the initial term.

3. The GMP, GMF, and valuation net premium based on the implied guaranteed rate.

b) The **valuation date** ( $t = t$ ) calculations are as follows:

1. The implied guaranteed rate for the remainder of the current period, using the **option cost** based on the market conditions at the valuation date.

2. The implied guaranteed rate for the period following the current period, based on the option cost on the valuation date.

3. A **re-projection of future guaranteed benefits** based on the implied valuation date.

4. The **present value** of the re-projected future guaranteed benefits.

Note that the GMP, GMF, and valuation net premium remain the same as calculated at issue ( $t = 0$ ).

#### XXXX16.4.4 Contracts with Secondary Guarantees

NAIC Actuarial Guideline 38 (AG 38) for reserves for UL products with secondary guarantees have 9 steps as follows:

1. The **minimum gross premium** required to satisfy the secondary guarantees is derived at issue ( $t = 0$ ) of the contract; the value of this premium will depend on whether the *stipulated premium* or the *shadow fund* method is in use. Its calculation uses the *policy charges* and *credited interest rate* guaranteed in the contract.

2. The **basic and deficiency reserves** for the secondary guarantees are calculated using the *minimum gross premium* described in Step (1).

3. The amount of **actual contributions** made in **excess of the minimum gross premiums** is determined, again with the process depending on whether the *stipulated premium* method or the *shadow fund* method is used.

4. At the valuation date,  $t$ , a determination is made regarding **amounts needed to fully fund** the secondary guarantee.

(a) Under the **shadow fund** method, this would be the *amount of the shadow fund account* needed to fully fund the guarantee.

(b) Under contracts not using the shadow fund method, this would be the amount of cumulative premiums paid in excess of the required level such that no future premiums are required to fully fund the guarantee.

Special rules apply to policies for which the secondary guarantee cannot be fully funded in advance. Here a prefunding ratio,  $r$ , ( $r \leq 1$ ), is calculated that measures the level of prefunding for the secondary guarantee, and is eventually used in the calculation of reserves. It is defined as

$$r = \frac{\text{Excess Payment}}{\text{Net Single Premium Required to Fully Fund the Guarantee}} \quad (16.7)$$

5. At the valuation date,  $t$ , the **net single premium** for the secondary guarantee coverage for the remainder of the secondary guarantee period is computed.  $NSP_t$ .
6. A net amount of **additional premiums** is determined by multiplying the prefunding ratio described in Step (4) times the difference between the net single premium of Step (5) and the basic plus deficiency (if any) reserve of Step (2).  $r(NSP_t - bpd_t V)$
7. A **reduced deficiency reserve** is determined by multiplying the deficiency reserve (if any) by the complement of the pre-funding ratio from Step (4).  $red\ d_t V = (1 - r)_{d_t} V$
8. Then the **actual reserve** is the lesser of (a) the net single premium of Step (5), or (b) the amount in Step (6) plus the basic and deficiency (if any) reserve from Step (2). This result might be reduced by applicable policy surrender charges.  $actual_t V = \min(NSP_t, Step6 + Step2)$
9. An **increased basic reserve** is computed by subtracting the reduced deficiency reserve of Step (7) from the reserve computed in Step (8), which then becomes the basic reserve.  $incr\ b_t V = actual_t V - red\ d_t V$ .  
Also  $AG38_t V = actual_t V + red\ d_t V$

## ACTEX MLC Chapter 14 Study Manual Vol II

## Universal Life Insurance

### 14.1 Basic Policy Design

**Account Value Accumulation**  $AV_t = (AV_{t-1} + P_t - EC_t - CoI_t)(1 + i_t^c)$

ULI with **variable failure benefit** (11.25c)  $AV_{t+1} = [AV_t + G_{t+1}(1 - r_{t+1}) - e_{t+1} - v_{t+1}q_{x+1}.B](1 + i_{t+1})$

ULI with **fixed failure benefit** (11.26b)  $AV_{t+1} = [AV_t + G_{t+1}(1 - r_{t+1}) - e_{t+1}](1 + i_{t+1}) - q_{x+1}.(B - AV_{t+1})$

$$(11.27) AV_{t+1} = \frac{[AV_t + G_{t+1}(1 - r_{t+1}) - e_{t+1}](1 + i_{t+1}) - q_{x+1}.B}{p_{x+t}}$$

### 14.2 Cost of Insurance and Surrender Value

#### Total Death Benefit

Specified Amount (Type A):  $\max(FA, f_t AV_t)$

Specified Amount plus the Account Value (Type B):  $\max(AV_t + X, f_t AV_t)$

#### Additional Death Benefit

Specified Amount (Type A):  $ADB_t = \max(FA - AV_t, (f_t - 1) AV_t)$

Specified Amount plus the Account Value (Type B):  $ADB_t = \max(X, (f_t - 1) AV_t)$

**General Formula for the cost of Insurance**  $CoI_t = q_t^* \times v_q \times ADB_t$

where  $CoI_t$  is the **cost of insurance** for the  $t^{th}$  time period, deducted from the account value at time  $t - 1$ ,

$q_t^*$  is the death probability (for the  $t^{th}$  time period) used to calculate the cost of insurance,

$v_q$  is the discount factor for discounting the cost of insurance to time  $t - 1$ ,

$ADB_t$  is the additional death benefit at time  $t$ .

#### Cost of Insurance for a Specified Amount (Type A) Policy

$$CoI_t = \max(CoI_t^{CF}, CoI_t^{wo\ CF})$$

where  $CoI_t^f = \frac{q_t^* v_q (FA - (AV_{t-1} + P - EC_t)(1 + i_t^c))}{1 - q_t^* v_q (1 + i_t^c)}$

$$CoI = \frac{q_t^* v_q (1 + i_t^c) (f_t - 1) (AV_{t-1} + P - EC_t)}{1 + q_t^* v_q (1 + i_t^c) (f_t - 1)}$$

#### Cost of Insurance for a Specified Amount plus the Account Value (Type B) Policy

$$CoI_t = \max(CoI_t^{CF}, CoI_t^{wo\ CF})$$

where  $CoI_t^{CF} = q_t^* v_q X$  and  $CoI_t^c = \frac{q_t^* v_q (1 + i_t^c) (f_t - 1) (AV_{t-1} + P - EC_t)}{1 + q_t^* v_q (1 + i_t^c) (f_t - 1)}$

### XXXX14.5 Profit Testing

**Expected Profit:**  $Pr_t = (AV_{t-1} + P - EC_t)(1 + i_t^e) - EDB_t - ESB_t - EAV_t$

**17.2 Deferred Annuity Products**

*17.2.2 Variable Deferred Annuity*

Investment Advisory Fee:  $IAF_t(n) = FV_{t-1}(n) \cdot \left[ (1 + IAF_t(n))^{1/365} - 1 \right]$ . (17.1)

Net Investment Rate for day  $t$ :  $NIF_t(n) = \frac{NII_t(n) - IAF_t(n) + RCG_t(n) + UCG_t(n)}{FV_{t-1}(n)}$ , (17.2)

Net Investment Factor:  $NIF_t(n) = 1 + NIR_t(n)$ . (17.3)

Sub-account  $n$  Fund Value recursion:  $FV_t(n) = FV_{t-1}(n) \cdot NIF_t(n) - EXP_t(n)$ , (17.4)

Overall contract Account Value on day  $t$ :  $AV_t = \sum_n FV_t(n)$ . (17.5)

*17.2.3 Equity -Indexed Deferred Annuity*

a) point-to-point:  $i_P = \frac{\text{Index value on closing day of index period}}{\text{Index value on initial day of index period}} - 1$ , (17.6)

b) monthly ave:  $i_{MA} = \frac{\frac{1}{12n} [\text{Sum of index values on the last day of each month during index period}]}{\text{Index value on initial day of index period}} - 1$ . (17.7)

c) with ratcheting  $i_P = \frac{\text{Index value on closing day } t \text{ of index period}}{\text{Index value on day } t - 1 \text{ of index period}} - 1$ ,

d) with ratcheting  $i_{MA} = \frac{\frac{1}{12n} [\text{Sum of index values on the last day of each month during index period}]}{\text{Index value on previous day } t - 1 \text{ of index period}} - 1$ .

**17.3 Immediate Annuity Products**

*17.3.2 Variable Immediate Annuity*  $APU = \frac{P_1}{PUV_1}$ . (17.8)

$PUV_t = PUV_{t-1} \left( \frac{NIF_t}{1 + AIR} \right)$ , (17.9)  $P_t = (APU) (PUV_t)$ . (17.10)

$P_t = (APU) (PUV_{t-1}) \left( \frac{NIF_t}{1 + AIR} \right) = P_{t-1} \left( \frac{NIF_t}{1 + AIR} \right)$ . (17.11)