**KING FAHD UNIVERSITY OF PETROLEUM AND MINERALS**

**Department of Mathematics and Statistics**

# Math 302 Final Exam

**Semester (163)**

**Aug 21, 2017 at 07:00-10:00 PM**

**Name:** ..................KEY..................

**I.D:** .................................... **Serial #:** .........

## Instructions

1. No electronic device (such as calculator, mobile phone, smart watch) is allowed in this exam.
2. Justify your answers. No credit is given for (correct) answers not supported by work.

| Question | Points |
|---|---|
| 1 | /20 |
| 2 | /20 |
| 3 | /20 |
| 4 | /20 |
| 5 | /30 |
| 6 | /30 |
| Total | /140 |

## Question 1 (20 points)

Consider the matrix $\mathbf{A} = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$.

Find the eigenvalues and the eigenvectors of $\mathbf{A}^{-1}$.

$$\det(A - \lambda I_2) = \begin{vmatrix} 2-\lambda & -1 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 + 1 = 0$$

$$\Rightarrow \lambda_1 = 2+i, \quad \lambda_2 = 2-i$$ 4

For $\lambda_1 = 2+i$,

$$(A - \lambda_1 I) = \begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix} \rightarrow \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & i \\ 0 & 0 \end{pmatrix}$$

So, if $(A - \lambda_1 I)K = 0 \Rightarrow k_1 = t, \quad k_2 = -it$

$$K = \begin{pmatrix} -it \\ t \end{pmatrix} = t \begin{pmatrix} -i \\ 1 \end{pmatrix}.$$

For $t = i \Longrightarrow K_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$. 4

For $\lambda_2 = 2-i$, we have

$$K_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$$ 2

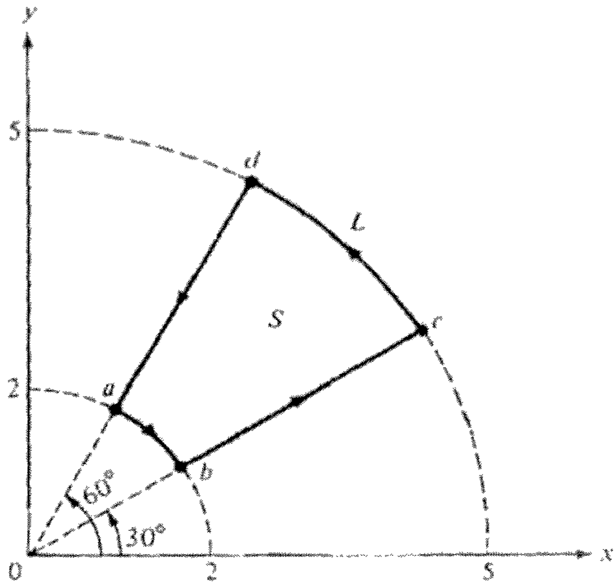
Thus, for $A^{-1}$, we have

$$\lambda_1 = \frac{1}{2+i} = \frac{2-i}{5} = \frac{1}{5}(2-i) \text{ and } K_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$$ 4

$$\lambda_2 = \frac{1}{2-i} = \frac{2+i}{5} = \frac{1}{5}(2+i) \text{ and } K_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$$ 4

**Question 2** (20 points)

Use Stokes's theorem to evaluate $\oint \mathbf{A}.d\mathbf{l}$ for $\mathbf{A} = \rho\cos\phi\ \mathbf{a}_\rho + \sin\phi\ \mathbf{a}_\phi$ around the path shown in the adjacent figure.




Curl $A = \nabla \times A$

$$= \frac{1}{\rho}\begin{vmatrix} a_\rho & \rho a_\phi & a_z \\ \partial\rho & \partial\phi & \partial z \\ \rho\cos\phi & \rho\sin\phi & 0 \end{vmatrix}$$

$$= \frac{1}{\rho}\left[(0-0)a_\rho - (0-0)\rho a_\phi + (1+\rho\sin\phi)a_z\right]$$

$$= \frac{1}{\rho}(1+\rho)\sin\phi\ a_z .$$ 6

Using Stokes's thm,

4

$$\oint A.dI = \int_S (\nabla \times A)\cdot dS \quad , \text{where } dS = \rho\, d\phi\, d\rho\, a_z$$

$$= \int_{30°}^{60}\int_2^5 \frac{1}{\rho}(1+\rho)\sin\phi\ \rho\, d\rho\, d\phi$$ 5

$$= \int_{30°}^{60°}\sin\phi\, d\phi \int_2^5 (1+\rho)\,d\rho$$

$$= -\cos\phi\Big|_{30°}^{60°}\left(\rho+\frac{\rho^2}{2}\right)\Big|_2^5$$ 5

$$= \frac{27}{4}(\sqrt{3}-1)$$

## Question 3 (10+10 points)

a. Given the potential $\mathbf{V} = \frac{10}{r^2}\sin\theta\cos\phi$. Calculate the work done in moving a 10 $\mu$C charge from point $A(2, 30^\circ, 120^\circ)$ to $B(4, 150^\circ, 60^\circ)$.

$$W = -Q\int_A^B E \cdot dl = Q V_{AB}$$

$$= Q(V_B - V_A)$$

$$= 10\left(\frac{10}{16}\sin 150^\circ \cos 60^\circ - \frac{10}{4}\sin 30^\circ \cos 120^\circ\right) \cdot 10^{-6}$$

$$= 10\left(\frac{10}{16}\cdot\frac{1}{2}\cdot\frac{1}{2} - \frac{10}{4}\cdot\frac{1}{2}\cdot\frac{-1}{2}\right)\cdot 10^{-6}$$

$$= 10\left(\frac{5}{32} + \frac{5}{8}\right)10^{-6} = \frac{125}{16}\mu J.$$

5

b. Find all values of $z$ such that $(\sin z + \cos z)^2 = 3$.

$$\cos^2 z + \sin^2 z + 2\sin z\cos z = 3$$

$$2\sin z\cos z = 2$$

$$\sin 2z = 1$$

$$\frac{e^{2iz} - e^{-2iz}}{2i} = 2 \implies e^{2iz} - e^{-2iz} - 4i = 0$$

4

Multiply by $e^{2iz}$: $e^{4iz} - 4ie^{2iz} - 1 = 0$

$$\implies e^{2iz} = \frac{4i \mp \sqrt{-16+4}}{2} = \frac{4i \mp 2\sqrt{3}i}{2} = (2\mp\sqrt{3})i$$

2

$$2iz = \ln(2\mp\sqrt{3})i = \log_e|2\mp\sqrt{3}| + i\left(\frac{\pi}{2} + 2n\pi\right)$$

2

$$\implies z = \left(\frac{\pi}{4} + n\pi\right) - \frac{i}{2}\log_e|2\mp\sqrt{3}| \quad , \quad n \text{ integer}$$

2

**Question 4** (10+10 points)

a. Consider $f(z) = (2x - y^2) + i(x^2 + 2y)$.

(1) Where $f(z)$ is differentiable? (2) Why $f(z)$ is ***nowhere*** analytic?

(1) $u(x,y) = 2x - y^2$ $\quad v(x,y) = x^2 + 2y$

$u_x = 2, \quad v_y = 2$

$u_y = -2y, \quad v_x = 2x$ (3)

C-R equations are satified for $y = x$

So, $f(z)$ is differentiable for any point on the line $y = x$. (3)

(2) For any point $z$ on the line $y=z$, there is no neighborhood or open disk about $z$ for which $f$ is diff. Thus, $f$ is nowhere analytic

(4)

b. Find all zeros of $f(z) = \sinh z$.

Sln $\sinh z = \sinh(x+iy)$

$= \sinh x \cos y + i \cosh x \sin y = 0$ (3)

$\Rightarrow \sinh x \cos y = 0$ and $\cosh x \sin y = 0$

$x = 0$ and $\sin y = 0$

$\Rightarrow$ or $y = n\pi$ (3)

$z = 0 + n\pi i = n\pi i, \quad n = 0, \mp 1, \mp 2, \ldots$

are the zeros.

(4)

**Question 5** (15+15 points)

a. Use Cauchy's integral formula to evaluate

$$\oint_C \frac{z^2}{z^2+9}\,dz, \text{ where } C \text{ is the circle } |z-2i|=4.$$

Only $z = 3i$ inside the contour $C$.

So, we write $\dfrac{z^2}{z^2+9} = \dfrac{\frac{z^2}{z+3i}}{z-3i}$ (5)

From C. I. formula

$$\oint \frac{z^2}{z^2+9}\,dz = \oint \frac{\frac{z^2}{z+3i}}{z-3i}\,dz = 2\pi i\, f(3i) \quad (5)$$

$$= 2\pi i\left(\frac{-9}{6i}\right) = -3\pi \quad (5)$$

b. Expand $f(z) = \frac{1}{(z-1)^2(z-3)}$ in a Laurent series valid for $0 < |z-1| < 2$ and find $Res(f(z), 1)$.

$$f(z) = \frac{1}{(z-1)^2}\cdot\frac{1}{-2+(z-1)} = \frac{-1}{2(z-1)^2}\cdot\frac{1}{1-\frac{z-1}{2}} \quad (4)$$

$$= \frac{-1}{2(z-1)^2}\left[1+\frac{z-1}{2}+\frac{(z-1)^2}{2^2}+\frac{(z-1)^3}{2^3}+\cdots\right]$$

valid for $0 < \frac{|z-1|}{2} < 1$ (4)

or $0 < |z-1| < 2$

Thus,

$$f(z) = \frac{-1}{2(z-1)^2} - \frac{1}{4(z-1)} - \frac{1}{8} - \frac{1}{16}(z-1) - \cdots \quad (3)$$

From this series, $Res(f, 1) = \frac{-1}{4}$. (4)

**Question 6** (15+15 points)

a. Use the Residue Theorem to evaluate

$$\oint_C \frac{e^z}{z^3+3z^2}\,dz, \text{ where the contour C is the circle } |z| = 2.$$

$z^3+3z^2 = z^2(z+3)$ has two zeros: $(-3)$ out of $C$,

and $z=0$ of order 2.

Thus, the integrand has a pole of order 2. (5)

$$\oint_C \frac{e^z}{z^3+3z^2}\,dz = 2\pi i \operatorname{Res}(f,0)$$

$$= 2\pi i \frac{1}{1!} \lim_{z\to 0} \frac{d}{dz}\left[z^2 \frac{e^z}{z^2(z+3)}\right] \quad (5)$$

$$= 2\pi i \lim_{z\to 0}\left[\frac{e^z(z+2)}{(z+3)^2}\right] = 2\pi i\left(\frac{2}{9}\right) = \frac{4\pi}{9} i$$

(5)

b. Evaluate the real integral $\int_0^{2\pi} \frac{d\theta}{10-6\cos\theta}$ by converting into a complex integral with unit circle contour.

$C: \; z = \cos\theta + i\sin\theta \;, \; 0 \le \theta \le 2\pi$

$$d\theta = \frac{dz}{iz}, \qquad \cos\theta = \frac{1}{2}(z+z^{-1}) \quad (2)$$

$$\int_0^{2\pi} \frac{d\theta}{10-6\cos\theta} = \frac{1}{i}\oint_C \frac{dz}{z\left(10-6\frac{z+z^{-1}}{2}\right)} = \frac{-1}{i}\oint_C \frac{dz}{3z^2-10z+3} \quad (4)$$

$$= \frac{i}{3}\oint_C \frac{dz}{(z-\frac{1}{3})(z-3)}, \quad \text{only } z=\tfrac{1}{3} \text{ in } C.$$

$$\Rightarrow \int_0^{2\pi} \frac{d\theta}{10-6\cos\theta} = \frac{i}{3}\cdot 2\pi i \operatorname{Res}\left(f, \tfrac{1}{3}\right) \quad (4)$$

$$= -\frac{2\pi}{3}\left(\lim_{z\to\frac{1}{3}} \frac{1}{z-3}\right) = -\frac{2\pi}{3}\cdot\frac{-3}{8} = \frac{\pi}{4}$$

(5)