KING FAHD UNIVERSITY OF PETROLEUM AND MINERALS

Department of Mathematics and Statistics

Math 302 Second Major Exam

Semester (163)

Aug 10, 2017 at 11:00-01:00 pm

Name: ....KEY........

I.D: .... Section: .... Serial #: ....

Instructions

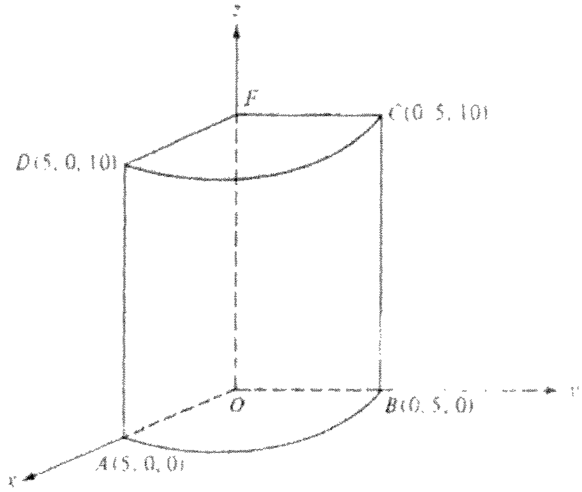
1. No electronic device (such as calculator, mobile phone, smart watch) is allowed in this exam.
2. Justify your answers. No credit is given for (correct) answers not supported by work.

Question	Points
1	/20
2	/20
3	/20
4	/20
5	/20
Total	/100

Question 1 (5+5+10 points)

Consider the shape in the adjacent figure. Calculate:

a. the length of ***CD***.



$CD = \int A \cdot dL \;,\; \rho = 5 \;,\; \phi = 0 \to \frac{\pi}{2}$

$z = 10$

$CD = \int_0^{\pi/2} \rho \, d\phi = \frac{5\pi}{2}$

3 2

b. the surface area of ***ABCD***.

$ABCD = \int\int_S A \cdot dS$ $\quad \rho = 5 \;,\; z = 0 \to 10$

$\phi = 0 \to \pi/2$

$= \int_0^{10} \int_0^{\pi/2} \rho \, d\phi \, dz$ 3

$= 5 \int_0^{10} \phi \Big|_0^{\pi/2} dz = \int_0^{10} 5\left(\frac{\pi}{2}\right) dz = 25\pi$ 2

c. the volume of ***ABDCFO***.

$V = \int\int\int_V dV = \int_0^{10} \int_0^{\pi/2} \int_0^{5} \rho \, d\rho \, d\phi \, dz$ 3

$= \int_0^{10} \int_0^{\pi/2} \frac{\rho^2}{2} \Big|_0^{5} d\phi \, dz$ 2

$= \frac{125}{2}\pi$

Question 2 (10+10 points)

a. Given $\boldsymbol{W} = x^2y^2 + xyz$. Compute the directional derivative of $\boldsymbol{W}$ in the direction of $\mathbf{v} = 4\mathbf{a}_x + 12\mathbf{a}_y + 3\mathbf{a}_z$ at $(2,0,-1)$.

$$\frac{dW}{dL} = \nabla W(P) \cdot \vec{u}$$

$$\nabla W = (2xy^2 + yz)\, a_x + (2x^2y + xz)\, a_y + (xy)\, a_z \quad 2$$

$$\nabla W(P) = -2\, a_y = \langle 0, -2, 0 \rangle \quad 2$$

$$\vec{u} = \frac{\langle 4, 12, 13 \rangle}{\sqrt{169}} = \frac{1}{13} \langle 4, 12, 3 \rangle \quad 2$$

$$\frac{dW}{dL} = \langle 0, -2, 0 \rangle \cdot \langle 4, 12, 13 \rangle \frac{1}{13} = \frac{-24}{13}$$

2 2

b. A point charge of 5 nC is located at the origin. If $V = 2\text{V}$ at the point $P(0,6,-8)$, find the electric potential at $B(1,5,7)$.

$$V_B - V_A = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r_B} - \frac{1}{r_P}\right) \quad 2$$

$$r_B = |(0,0,0) - (1,5,7)| = \sqrt{1+25+49} = \sqrt{75} \quad 2$$

$$r_P = |(0,0,0) - (0,6,8| = \sqrt{100} = 10 \quad 2$$

$$V_B - 2 = \frac{5 \times 10^{-9}}{4\pi\left(\frac{10^{-9}}{36\pi}\right)}\left(\frac{1}{\sqrt{75}} - \frac{1}{10}\right) \quad 2$$

$$= 45\left(\frac{10 - \sqrt{75}}{10\sqrt{75}}\right) + 2 \quad 2$$

Question 3 (10+10 points)

Consider $\mathbf{D} = \rho^2 \cos^2 \phi \, \mathbf{a}_\rho + z \sin \phi \, \mathbf{a}_\phi$.

a. Find div $\mathbf{D}$.

$$div\, D = \nabla . D = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho D_\rho) + \frac{1}{\rho}\frac{\partial D_\phi}{\partial \phi} + \frac{\partial D_z}{\partial z} \quad 3$$

$$= \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho^3 \cos^2 \phi) + \frac{1}{\rho}(z \cos\phi) + 0 \quad 5$$

$$= 3\rho \cos^2\phi + \frac{1}{\rho}(z \cos\phi) \quad 2$$

b. Use the divergence theorem to calculate the flux of **D** over the closed surface of the cylinder $0 \le z \le 1, \ \rho = 4$.

D. theorem $\oint\!\!\oint_S D \cdot ds = \int\!\!\int\!\!\int_v \nabla . D \, dv$ 2

$$\text{flux} = \int_0^1 \int_0^{2\pi} \int_0^4 \left(3\rho \cos^2\phi + \frac{1}{\rho}(z \cos\phi)\right)(\rho \, d\rho \, d\phi \, dz) \quad 5$$

$$= \int_0^1 \int_0^{2\pi} (64 \cos^2\phi + 4z \cos\phi) \, d\phi \, dz \quad 2$$

$$= \int_0^1 64(\pi) \, dz$$

$$= 64\pi \quad 1$$

Question 4 (5+10+5 points)

Consider $\mathbf{B} = (y + z\cos xz)\mathbf{a}_x + x\mathbf{a}_y + x\cos xz\ \mathbf{a}_z$.

a. Show that **B** is conservative.

$$\text{Curl } B = \nabla \times B = \begin{vmatrix} a_x & a_y & a_z \\ \partial x & \partial y & \partial z \\ y + z\cos xz & x & x\cos xz \end{vmatrix} \quad (2)$$

$$= (0-0)a_x - (\cos xz - xz\sin xz - \cos xz + xz\sin xz)a_y + (1-1)a_z = 0 \quad (3)$$

$\Rightarrow$ Conservative.

b. Find the potential function ϕ of **B**.

$\phi_x = y + z\cos xz$ (2)

$\phi = yx + \sin xz + g(y,z)$ (2)

$\phi_y = x + g_y(y,z) = x \Rightarrow g_y(y,z) = 0$ (2)

$\phi = yx + \sin xz + c$ (4)

c. Evaluate $\int_{(1,0,\pi)}^{(2,1,\pi)} \boldsymbol{B}.\,d\boldsymbol{l}$.

$$\int_{(1,0,\pi)}^{(2,1,\pi)} B \cdot dl = \phi(2,1,\pi) - \phi(1,0,\pi) \quad (3)$$

$$= 2 \quad (2)$$

Question 5 (10+10 points)

a. Let $z = 1 - i$. Calculate z^8.

$|z| = \sqrt{1+1} = \sqrt{2}$ 4

$\theta = \tan^{-1} \frac{y}{x} = \tan^{-1}(-1) = -\frac{\pi}{4}$

$z^8 = (\sqrt{2})^8 \left(\cos 8\left(\frac{-\pi}{4}\right) + i \sin 8\left(\frac{-\pi}{4}\right) \right)$ 2 2

$= 16$

2

b. Sketch the complex region that describes the intersection between

$$|z - 4 - 2i| < 3 \quad \text{and} \quad 2 \le \text{Re}(z) \le 5.$$