# KING FAHD UNIVERSITY OF PETROLEUM AND MINERALS

## Department of Mathematics and Statistics

## Math 302 First Major Exam

## Semester (163)

## July 26, 2017 at 09:00-11:00 PM

**Name:** ......................KEY..............................................

**I.D:** .................................... **Section:** ........ **Serial #:** .........

## Instructions

1. No electronic device (such as calculator, mobile phone, smart watch) is allowed in this exam.
2. Justify your answers. No credit is given for (correct) answers not supported by work.

| Question | Points |
|---|---|
| 1 | /15 |
| 2 | /15 |
| 3 | /20 |
| 4 | /15 |
| 5 | /20 |
| 6 | /15 |
| Total | /100 |

## Question 1 (5+5+5 points)

Consider the set $E = \{(a, b, c) \in \mathbb{R}^3 | 2a - b + c = 0\}$

a. Show that $E$ is a subspace of $\mathbb{R}^3$.

Let $u_1 = \langle a_1, b_1, c_1 \rangle$ , $u_2 = \langle a_2, b_2, c_2 \rangle \in E$

$\Rightarrow 2a_1 - b_1 + c_1 = 2a_2 - b_2 + c_2 = 0$

$\Rightarrow 2(a_1 + a_2) - (b_1 + b_2) + (c_1 + c_2)$

$= (2a_1 - b_1 + c_1) + (2a_2 - b_2 + c_2) = 0 + 0 = 0$ (3)

$\Rightarrow u_1 + u_2 \in E$

For any $k \in \mathbb{R}$, $k(2a) - kb + kc = k(2a - b + c) = 0$

$\Rightarrow ku_1 \in E$ $\Rightarrow$ $E$ is a subspace from $\mathbb{R}^3$ (2)

b. Find a basis and dimension of $E$.

Let $u = \langle a, b, c \rangle \in E$ then

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ b \\ b - 2a \end{pmatrix} = \begin{pmatrix} a \\ 0 \\ -2a \end{pmatrix} + \begin{pmatrix} 0 \\ b \\ b \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$ (2)

as $u_1 = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$, $u_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ are independent and span $E$ (1)

$\Rightarrow \{u_1, u_2\}$ forms a basis for $E$. (1)

The dim $= 2$

c. Write $v = (1,6,4) \in E$ as a linear combination of vectors in that basis.

$$v = \begin{pmatrix} 1 \\ 6 \\ 4 \end{pmatrix} = 1 \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} + 6 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$ by (b). (4)

Thus $v = 1u_1 + 6u_2$. (1)

## Question 2 (3+3+3+3+3 points)

Let $\mathbf{A} = \begin{bmatrix} 2 & 0 & 1 \\ -2 & 3 & 4 \\ -5 & 5 & 6 \end{bmatrix}$, $\mathbf{A}^{-1} = \begin{bmatrix} -2 & 5 & -3 \\ -8 & 17 & -10 \\ 5 & -10 & k \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix}$.

a. Find $k$.

Multiply the Last Row of $A$ and last Column of $A^{-1}$ gives

$+15 + 50 + 6k = 1 \Rightarrow 6k = 36 \Rightarrow k = 6$

(2) (1)

b. Find $(\mathbf{A}^T)^{-1}$.

$$(A^T)^{-1} = (A^{-1})^T = \begin{bmatrix} -2 & -8 & 5 \\ 5 & 17 & -10 \\ -3 & -10 & 6 \end{bmatrix}$$

(2) (1)

c. What is the rank of $\mathbf{A}$ and the rank of $[\mathbf{A}|\mathbf{B}]$? Explain.

As $A$ is nonsingular $\Rightarrow$ Rank $A = 3$ (2)

Hence, Rank $[A|B] = 3$ (1)

d. Solve the system $\mathbf{AX} = \mathbf{B}$.

$$X = A^{-1}B = \begin{bmatrix} -2 & 5 & -3 \\ -8 & 17 & -10 \\ 5 & -10 & 6 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 21 \\ 72 \\ -43 \end{bmatrix}$$

(2) (1)

e. What type of solutions the homogenous system $\mathbf{AX} = \mathbf{O}$ has? where $\mathbf{O}$ is the zero $3 \times 1$ matrix.

Trivial solution because $A$ is nonsingular.

(2) (1)

**Question 3** (10+10 points)

Consider the matrix $\mathbf{A} = \begin{bmatrix} 3 & -2 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 3 \end{bmatrix}$.

Eigenvalues of $\mathbf{A}$ are $\lambda_1 = 1,\ \lambda_2 = 3,\ \lambda_3 = 5$ and the corresponding eigenvectors are $X_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$, $X_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$, $X_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, respectively.

a. Find $a + b + c$.

$$(A - \lambda I) = (A - 5I) = \begin{bmatrix} -2 & -2 & 0 \\ -1 & -2 & -2 \\ 0 & -1 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$ (2)

$$\rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$ (3)

Let $c = t \Rightarrow b = -2t$ and $a = 2t$, for any $t \neq 0$ (3)

Hence $X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$ and $a + b + c = 1$ (1)

b. Find the eigenvalues and eigenvectors for $\boldsymbol{A}^{-1}$.

$\lambda_1 = 1$ and $X_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$ (3)

$\lambda_2 = \frac{1}{3}$ and $X_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$ (3)

$\lambda_3 = \frac{1}{5}$ and $X_3 = \begin{bmatrix} +2 \\ -2 \\ 1 \end{bmatrix}$ (3)

As $A^{-1}$ has the reciprocal of eigenvalues and the (1) same eigenvectors of $A$.

## Question 4 (15 points)

Determine whether the given matrix $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ is diagonalizable. If so, find the matrix $\mathbf{P}$ that diagonalizes $\mathbf{A}$ and the diagonal matrix $\mathbf{D}$ such that $\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP}$.

To find the eigenvalues, we solve

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0 \quad (2)$$

$\Rightarrow \lambda_1 = i$ and $\lambda_2 = -i$ $\Rightarrow$ Two distinct eigenvalues $\Rightarrow$ $A$ is diagonalizable (2)

To find the eigenvectors,

$$(A - \lambda I) = \begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} \to \begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix} \to \begin{pmatrix} i & -1 \\ 0 & 0 \end{pmatrix}$$

$\Rightarrow X_1 = t\begin{pmatrix} -i \\ 1 \end{pmatrix}$ or $X_1 = \begin{pmatrix} -i \\ 1 \end{pmatrix}$ (2)

and hence, $X_2 = \bar{X}_1 = \begin{pmatrix} i \\ 1 \end{pmatrix}$ (2)

Thus

$$P = \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix} \quad (2)$$

and $D = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ (2)

## Question 5 (15+5 points)

Consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 0 \\ 7 & 0 & 1 \end{bmatrix}$.

Eigenvalues of $\mathbf{A}$ are $\lambda_1 = 1,\ \lambda_2 = -6,\ \lambda_3 = 8$ and the corresponding eigenvectors are $X_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $X_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $X_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, respectively.

a. Find an orthogonal matrix $\mathbf{P}$ that diagonalizes $\mathbf{A}$ and the diagonal matrix $\mathbf{D}$ such that $\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$.

$$P' = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

After normalizing these vectors, $P'$ becomes

$$P = \begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \quad (5), \qquad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 8 \end{bmatrix} \quad (5)$$

b. Find $\mathbf{P}^{-1}$ and verify the equation $\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$.

As $A$ is symmetric $\Rightarrow P^{-1} = P^{T} \Rightarrow$

$$P^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \quad (4)$$

$$P^{-1}AP = \begin{bmatrix} 0 & 1 & 0 \\ -6/\sqrt{2} & 0 & 6/\sqrt{2} \\ 8/\sqrt{2} & 0 & 8/\sqrt{2} \end{bmatrix} \begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

**Question 6** (6+9 points)

Given the point $P(-3,3\sqrt{3},2)$ and the vector $\mathbf{A} = y\mathbf{a}_x + (x+z)\mathbf{a}_y$.

a. Evaluate **A** at $P$ and find $|\mathbf{A}(x,y,z)|$ at $P$.

$$A(P) = 3\sqrt{3}\, a_x - 1\, a_y .$$ (3)

$$|A(P)| = \sqrt{(3\sqrt{3})^2 + (-1)^2}$$

$$= \sqrt{27+1} = \sqrt{28} = 2\sqrt{7}$$ (3)

b. Express $P$ **and** **A** in cylindrical coordinates. Evaluate **A** at $P$ **and** find $|\mathbf{A}(\rho,\phi,z)|$ at $P$.

$$* \ \rho = \sqrt{x^2+y^2} = \sqrt{9+27} = 6$$

$$\phi = \tan^{-1}\frac{y}{x} = \tan^{-1}(-\sqrt{3}) = 120^\circ , \quad z = 2$$

$$\Rightarrow P(-3, 3\sqrt{3}, 2) = P(6, 120^\circ, 2)$$ (3)

$$* \ A = \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y \\ x+z \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \rho\sin\phi \\ \rho\cos\phi + z \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \rho\cos\phi\sin\phi + \sin\phi(\rho\cos\phi + z) \\ -\rho\sin^2\phi + \cos\phi(\rho\cos\phi + z) \\ 0 \end{bmatrix}$$ (3)

$$A(6,120^\circ,2) = \begin{bmatrix} -6\sqrt{3}/4 - \sqrt{3}/2 \\ -18/4 + 1/2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2\sqrt{3} \\ -4 \\ 0 \end{bmatrix}$$ (2)

$$|A(6,120^\circ,2)| = \sqrt{(-2\sqrt{3})^2 + (-4)^2}$$

$$= \sqrt{12+16} = \sqrt{28} = 2\sqrt{7}$$ (1)