Exercise #1: (14pts) Consider the function f from R^3 to R defined by

$$f(x,y,z) = xy + e^{yz}$$

(a) Find the directional derivatives of f at the point $P_0(1,1,1)$ in the direction of the point $P_1(2,1,0)$.

(b) Find the direction in which the directional derivative has maximum value. What is this maximum value?

ANSWER:

(a) We have $\nabla f(x,y,z) =< \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} >=< y, x + ze^{yz}, ye^{yz} >$ so that $\nabla f(1,1,1) =< 1, 1+e, e >$. We have $\overrightarrow{P_0P_1} =< 1, 0, -1 >$ so that $||\overrightarrow{P_0P_1}|| = \sqrt{2}$. Thus $u = \frac{1}{||\overrightarrow{P_0P_1}||}\overrightarrow{P_0P_1} =< \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}} >$ is a unit vector along $\overrightarrow{P_0P_1}$. Thus the directional derivative of f at P_0 in the direction from P_0 to P_1 is

$$\begin{aligned} D_{\overrightarrow{P_0P_1}} f(P_0) &= D_u f(P_0) = \nabla f(P_0) \cdot u \\ &= < 1, 1+e, e > \cdot < \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}} > \\ &= \frac{1}{\sqrt{2}}(1-e) \end{aligned}$$

(b) The directional derivative has maximum value in the direction of the gradient $\nabla f(1,1,1) =< 1, 1+e, e >$ and this maximum value is $||\nabla f(1,1,1)|| = \sqrt{1^2 + (1+e)^2 + 1^2} = \sqrt{2 + 2e + 2e^2}$.

Exercise #2: (18pts) Consider the vector field

$$F(x,y,z) = y^2e^z\mathbf{i} + 2xye^z\mathbf{j} + xy^2e^z\mathbf{k}$$

(a) Is the vector field F conservative? if yes, find a potential $\varphi(x,y,z)$.

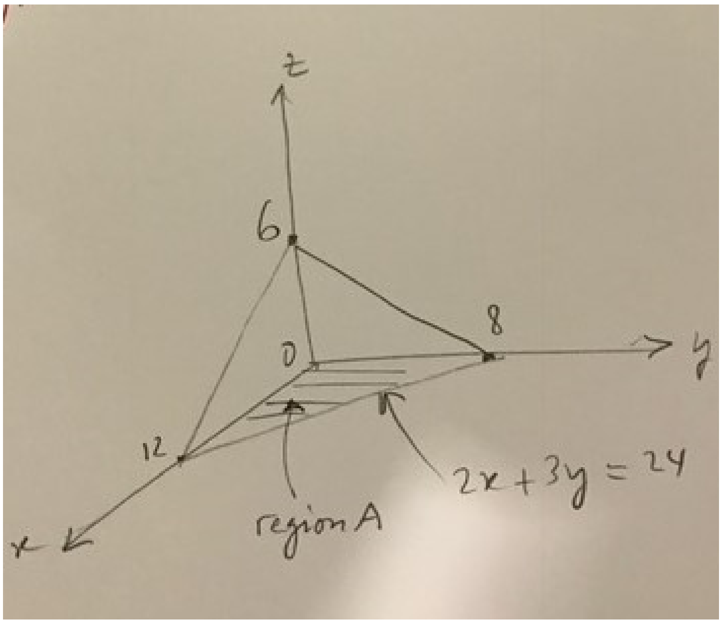
(b) Find the work done by F along the straight line from $P_0(1,1,0)$ to $P_1(1,1,1)$.

ANSWER:

(a) We have

$$\begin{aligned} \operatorname{curl} F &= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2e^z & 2xye^z & xy^2e^z \end{vmatrix} \\ &= (2xye^z - 2xye^z)\, i - \left(y^2e^z - y^2e^z\right) j + (2ye^z - 2ye^z)\, k = 0 \end{aligned}$$

Thus the vector field is conservative. So, there is a potential function φ such that $\nabla\varphi = F$ that is $\frac{\partial \varphi}{\partial x} = y^2e^z$, $\frac{\partial \varphi}{\partial y} = 2xye^z$ and $\frac{\partial \varphi}{\partial z} = xy^2e^z$. Starting from $\frac{\partial \varphi}{\partial x} = y^2e^z$ and integrating with respect to x we get $\varphi = xy^2e^z + g(y,z)$. Differentiating with respect to y, we get $\frac{\partial \varphi}{\partial y} = 2xye^z + g_y(y,z) \equiv 2xye^z$ so $g_y(y,z) \equiv 0$ which gives $g(y,z) \equiv h(z)$. Differentiating now with respect to z we get $\frac{\partial \varphi}{\partial z} = xy^2e^z + h'(z) \equiv xy^2e^z$ so that $h'(z) \equiv 0$ leading to $h(z) = c$. Hence a potential is given by $\varphi = xy^2e^z + c$.where c is an arbitrary constant.



(b) Since the vector field F is conservative the line integral is independent of path and the work done by F along the straight line C from $P_0(1,1,0)$ to $P_1(1,1,1)$is

$$\int_C F \cdot dR = \varphi(1,1,1) - \varphi(1,1,0) = e - 1$$

Exercise #3: (18pts) Find the surface integral

$$I = \int\int_S z dS$$

where the surface S is defined by the portion of the plane $2x + 3y + 4z = 24$, in the first octant $x \geq 0$, $y \geq 0$, $z \geq 0$.

ANSWER:
Since

$$z = 6 - \frac{1}{2}x - \frac{3}{4}y \text{ and } dS = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dA = \frac{\sqrt{29}}{4} dA$$

I becomes,

$$\begin{aligned} I &= \int\int_A \left(6 - \frac{1}{2}x - \frac{3}{4}y\right) \frac{\sqrt{29}}{4} dA \\ &= \frac{\sqrt{29}}{4} \int_0^{12} \int_0^{8-\frac{2}{3}x} \left(6 - \frac{1}{2}x - \frac{3}{4}y\right) dy dx \\ &= \frac{\sqrt{29}}{4} \int_0^{12} \left[6y - \frac{1}{2}xy - \frac{3}{8}y^2\right]_0^{8-\frac{2}{3}x} dx \\ &= \frac{\sqrt{29}}{4} \int_0^{12} \left[6\left(8 - \frac{2}{3}x\right) - \frac{1}{2}x\left(8 - \frac{2}{3}x\right) - \frac{3}{8}\left(8 - \frac{2}{3}x\right)^2\right] dx \\ &= 24\sqrt{29} \end{aligned}$$

Exercise #4: (22pts) Verify Green's theorem for the vector field $F(x, y) = x\mathbf{i} + xy^2\mathbf{j}$ and C is the curve $x^2 + y^2 = 4$ oriented counterclockwise.

ANSWER:

We have (i) The curve C is a closed (piecewise) continuous and oriented positively.(ii) The vector field F is continuous inside the region enclosed by C and in its neighborhood together with the partial derivatives $\frac{\partial Q}{\partial x}, \frac{\partial P}{\partial y}$ when $F = Pi + Qj$ so Green's Theorem is applicable and

$$\oint_C F \cdot dR = \int\int_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

On the one side we have,

$$\oint_C F \cdot dR = \oint_C xdx + xy^2 dy$$

Taking $x = 2\cos\theta$ and $y = 2\sin\theta$ we get $dxdy = rdrd\theta$ and

$$\begin{aligned}
\oint_C F \cdot dR &= \oint_C \left[-4\cos\theta\sin\theta + 16\cos^2\theta\sin^2\theta\right] d\theta \\
&= \left[-2\sin^2\theta + 2\left(\theta - \frac{\sin 4\theta}{4}\right)\right]_{\theta=0}^{\theta=2\pi} \\
&= 4\pi
\end{aligned}$$

On the other side we have,

$$\begin{aligned}
\int\int_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA &= \int\int_A y^2 dA \\
&= \int_0^{2\pi}\int_0^2 \left[r^2\sin^2\theta\right] rdrd\theta \\
&= \left[\frac{r^4}{4}\right]_{r=0}^{r=2} \int_0^{2\pi} \sin^2\theta d\theta \\
&= 4\int_0^{2\pi} \frac{1-\cos 2\theta}{2} d\theta = 4\pi
\end{aligned}$$

The sides are equal so Greens ' theorem has been verified.

Exercise #5: (14pts) Use Stokes' theorem to evaluate

$$\int\int_S (\operatorname{curl} F) \cdot ndS$$

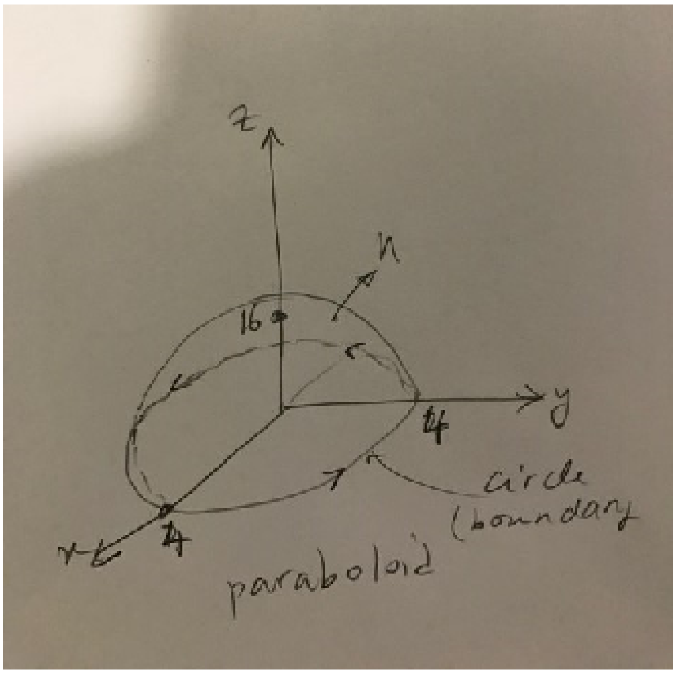
where $F(x, y, z) = y\mathbf{i} + zy^2\mathbf{j} + \cos(z)\mathbf{k}$ and S is the portion of the paraboloid $z = 16 - x^2 - y^2$, $z \geq 0$, oriented upward.

ANSWER:

We have,

(i) The surface is piecewise smooth, (ii) The vector field F is continuous together with the partial derivatives involved so the hypotheses of Stokes 'theorem are satisfied. Thus, Stokes 'theorem is applicable and

$$\oint_C F \cdot dR = \int\int_S (\operatorname{curl} F) \cdot ndS$$



Thus,

$$\int\int_S (\operatorname{curl} F) \cdot ndS = \oint_C F \cdot dR = \oint_C ydx + zy^2dy + \cos zdz$$

Since $z = 0$, the integral becomes

$$\int\int_S (\operatorname{curl} F) \cdot ndS = \oint_C ydx$$

A change to polar coordinates $x = 4\cos\theta$, $y = 4\sin\theta$ yields,

$$\int\int_S (\operatorname{curl} F) \cdot ndS = -16\int_0^{2\pi} \sin^2\theta d\theta = -16\int_0^{2\pi} \frac{1-\cos 2\theta}{2} d\theta = -8\left[\theta - \frac{\sin 2\theta}{2}\right] d\theta = -16\pi$$

Exercise #6: (14pts) Use the divergence theorem to evaluate

$$\int\int_S F \cdot ndS$$

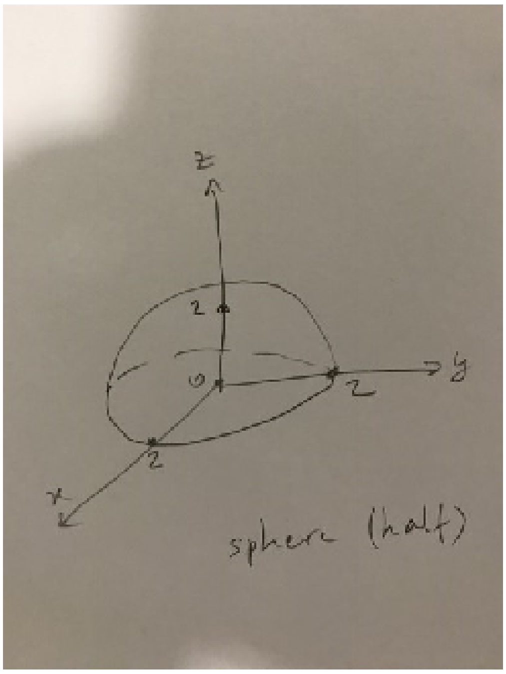
where $F(x, y, z) = xy^2\mathbf{i} + 2y\mathbf{j} - zy^2\mathbf{k}$ and S is the surface of the solid defined by $x^2+y^2+z^2 \leq 4$ and $z \geq 0.$, oriented outward.

ANSWER:

We have

(i) The surface is closed and bounded oriented outward and piecewise smooth. (ii) The vector field is continuous and the partial derivatives involved are continuous throught the a region containing the surface. So the hypotheses of the Divergence theorem are satisfied. Thus, the Divergence theorem is applicable and,

$$\int\int_S F \cdot ndS = \int\int\int_V \operatorname{div} FdV$$



Now, $\operatorname{div} F = y^2 + 2 - y^2 = 2$, hence,

$$\begin{aligned}\int\int_S F\cdot n dS &= 2\int\int\int_V dV \\ &= 2 \text{ times (half the volume of the sphere)} \\ &= \frac{4}{3}\pi r^3_{|r=2} = \frac{32}{3}\pi\end{aligned}$$